Liberal Arts Mathematics

Course No. 1208300

Bureau of Exceptional Education and Student Services
Florida Department of Education

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Curriculum Improvement Project
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LEON COUNTY SCHOOLS

Exceptional Student Education

http://www.leon.k12.fl.us/public/pass/

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Unit 1: Understanding Real Numbers

This unit emphasizes the relationships between sets of real numbers and the rules involved when working with them.

Unit Focus

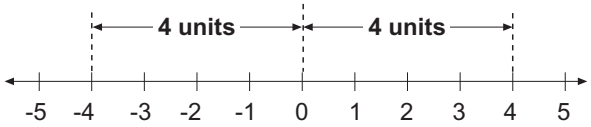
Number Sense, Concepts, and Operations

- Understand concrete and symbolic representations of real *and complex* numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)

Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

absolute value ..... a number's distance from zero (0) on a number line; distance expressed as a positive value
Example: The absolute value of both 4, written $|4|$, and negative 4, written $|-4|$, equals 4.



addend ..... any number being added
Example: In 14 + 6 = 20,
14 and 6 are addends.

additive identity ..... the number zero (0); when zero (0) is added to another number the sum is the number itself
Example: 5 + 0 = 5

additive inverses ..... a number and its opposite whose sum is zero (0); also called *opposites*
Example: In the equation 3 + -3 = 0,
3 and -3 are additive inverses, or *opposites*, of each other.

associative property ..... the way in which three or more numbers are grouped for addition or multiplication does *not* change their sum or product, respectively
Example: (5 + 6) + 9 = 5 + (6 + 9) *or*
(2 x 3) x 8 = 2 x (3 x 8)

commutative property ..... the order in which two numbers are added or multiplied does *not* change their sum or product, respectively
Example: $2 + 3 = 3 + 2$ *or* $4 \times 7 = 7 \times 4$

decimal number ..... any number written with a decimal point in the number
Example: A decimal number falls between two whole numbers, such as 1.5 falls between 1 and 2. Decimal numbers smaller than 1 are sometimes called decimal fractions, such as five-tenths is written 0.5.

denominator ..... the bottom number of a fraction, indicating the number of equal parts a whole was divided into
Example: In the fraction $\frac{2}{3}$ the denominator is 3, meaning the whole was divided into 3 equal parts.

digit ..... any one of the 10 symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9

element or **member** ..... one of the objects in a set

equation ..... a mathematical sentence in which two expressions are connected by an equality symbol
Example: $2x = 10$

equivalent
(forms of a number) ..... the same number expressed in different forms
Example: $\frac{3}{4}$, 0.75, and 75%

even integers ..... any integer divisible by 2
Example: {… -4, -2, 0, 2, 4 …}

exponent (exponential form) ..... the number of times the base occurs as a factor
Example: 2^3 is the exponential form of 2 x 2 x 2. The numeral two (2) is called the *base*, and the numeral three (3) is called the *exponent*.

expression ..... a collection of numbers, symbols, and/or operation signs that stands for a number
Example: $4r^2$; $3x + 2y$; $\sqrt{25}$
Expressions do *not* contain equality (=) or inequality (<, >, ≤, ≥, or ≠) symbols.

finite set ..... a set in which a whole number can be used to represent its number of elements; a set that has bounds and is limited

fraction ..... any part of a whole
Example: One-half written in fractional form is $\frac{1}{2}$.

grouping symbols ..... parentheses ( ), braces { }, brackets [ ], and fraction bars indicating grouping of terms in an expression

infinite set ..... a set that is not finite; a set that has no boundaries and no limits

integers ..... the numbers in the set
{… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

irrational number ..... a real number that cannot be expressed as a ratio of two integers
Example: $\sqrt{2}$

multiples ..... the numbers that result from multiplying a given whole number by the set of whole numbers
Example: The multiples of 15 are 0, 15, 30, 45, 60, 75, etc.

multiplicative identity ..... the number one (1); the product of a number and the multiplicative identity is the number itself
Example: 5 x 1 = 5

multiplicative inverse (reciprocals) ..... any two numbers with a product of 1
Example: 4 and $\frac{1}{4}$; zero (0) has no multiplicative inverse

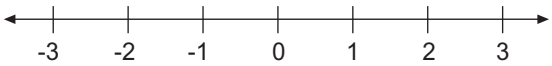
natural numbers (counting numbers) ..... the numbers in the set {1, 2, 3, 4, 5, …}

negative integers ..... integers less than zero

negative numbers ..... numbers less than zero

null set (ø) or **empty set** ..... a set with no elements or members

number line ..... a line on which ordered numbers can be written or visualized



numeral ..... a symbol which is not a variable, used to represent a number
Example: 25, $\sqrt{5}$, or 3.14

odd integers ..... any integer *not* divisible by 2
Example: {… -5, -3, -1, 1, 3, 5 …}

opposites ..... two numbers whose sum is zero
Example: $\underset{\text{opposites}}{-5 + 5} = 0$ *or* $\underset{\text{opposites}}{\frac{2}{3} + \left(-\frac{2}{3}\right)} = 0$

order of operations ..... the order of performing computations in parentheses first, then exponents or powers, followed by multiplication and/or division (as read from left to right), then addition and/or subtraction (as read from left to right); also called *algebraic order of operations*
Example: $5 + (12 - 2) \div 2 - 3 \times 2 =$
$5 + 10 \div 2 - 3 \times 2 =$
$5 + 5 - 6 =$
$10 - 6 =$
4

pattern (relationship) ..... a predictable or prescribed sequence of numbers, objects, etc.; may be described or presented using manipulatives, tables, graphics (pictures or drawings), or algebraic rules (functions)
Example: 2, 5, 8, 11 … is a pattern. Each number in this sequence is three more than the preceding number. Any number in this sequence can be described by the algebraic rule, $3n - 1$, by using the set of counting numbers for n.

percent (%) ..... a special-case ratio which compares numbers to 100 (the second term)
Example: 25% means the ratio of 25 to 100.

pi (π) ..... the symbol designating the ratio of the circumference of a circle to its diameter; an irrational number with common approximations of either 3.14 or $\frac{22}{7}$

positive integers ..... integers greater than zero

positive numbers ..... numbers greater than zero

power (of a number) ..... an exponent; the number that tells how many times a number is used as a factor
Example: In 2^3, 3 is the power.

product ..... the result of multiplying numbers together
Example: In $6 \times 8 = 48$,
48 is the product.

quotient ..... the result of dividing two numbers
Example: In $42 \div 7 = 6$,
6 is the quotient.

ratio ..... the comparison of two quantities
Example: The ratio of a and b is a:b or $\frac{a}{b}$,
where $b \neq 0$.

rational number ..... a real number that can be expressed as a ratio of two integers

real numbers ..... the set of all rational and irrational numbers

reciprocals ..... two numbers whose product is 1; also called *multiplicative inverses*
Example: Since $\frac{3}{4} \times \frac{4}{3} = 1$, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.

repeating decimal ..... a decimal in which one digit or a series of digits repeat endlessly

Example: $0.3333333\ldots$ or $0.\overline{3}$
$24.6666666\ldots$ or $24.\overline{6}$
$5.27272727\ldots$ or $5.\overline{27}$
$6.2835835\ldots$ or $6.2\overline{835}$

root ..... an equal factor of a number

Example:
In $\sqrt{144} = 12$, 12 is the square root.
In $\sqrt[3]{125} = 5$, 5 is the cube root.

rounded number ..... a number approximated to a specified place

Example: A commonly used rule to round a number is as follows.

- If the digit in the first place after the specified place is 5 or more, *round up* by adding 1 to the digit in the specified place ($\underline{4}61$ rounded to the nearest hundred is 500).
- If the digit in the first place after the specified place is less than 5, *round down* by *not* changing the digit in the specified place ($\underline{4}41$ rounded to the nearest hundred is 400).

scientific notation ..... a shorthand method of writing very large or very small numbers using exponents in which a number is expressed as the product of a power of 10 and a number that is greater than or equal to one (1) and less than 10

Example: $7.59 \times 10^5 = 759{,}000$

set ..... a collection of distinct objects or numbers

simplify an expression ..... to perform as many of the indicated operations as possible

solve ..... to find all numbers that make an equation or inequality true

sum ..... the result of adding numbers together
Example: In $6 + 8 = 14$,
14 is the sum.

terminating decimal ..... a decimal that contains a finite (limited) number of digits
Example: $\frac{3}{8} = 0.375$
$\frac{2}{5} = 0.4$

value (of a variable) ..... any of the numbers represented by the variable

variable ..... any symbol, usually a letter, which could represent a number

whole number ..... the numbers in the set $\{0, 1, 2, 3, 4, \ldots\}$

Unit 1: Understanding Real Numbers

Introduction

The focus of liberal arts mathematics is to strengthen and expand algebraic and geometry skills. These skills are necessary for further study and success in mathematics. Liberal arts mathematics reinforces

- an understanding of the real number system
- an understanding of different sets of numbers
- an understanding of various ways of representing numbers.

Many topics in this unit will be found again in later units. There is an emphasis on problem solving, estimation, and real-world applications.

Lesson One Purpose

- Understand concrete and symbolic representations of real *and complex* numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)

The Set of Real Numbers

A **set** is a collection. It can be a collection of DVDs, books, baseball cards, or even numbers. Each item in the *set* is called an **element** or **member** of the set. In algebra, we are most often interested in sets of numbers.

A set *can be a collection of books or numbers.*

The first set of numbers you learned when you were younger was the set of *counting numbers*, which are also called the **natural numbers**. These are the **positive numbers** you count with (1, 2, 3, 4, 5, …). Because this set has *no final number*, we call it an **infinite set**. A set that has a *specific number of elements* is called a **finite set**.

Mathematicians like to use symbols to represent sets. One type of **grouping symbols** are *braces*. Braces { } are the symbols we use to show that we are talking about a set.

A set with *no elements* or members is called a **null set (ø)** or **empty set**. It is often denoted by an *empty set* of braces { }.

The set of counting numbers looks like {1, 2, 3, …}.

Remember: The counting numbers can also be called the *natural numbers*, naturally!

The set of natural-number **multiples** of 10 is {10, 20, 30, …}

The set of **integers** that are *multiples* of 10 is
{… , -30, -20, -10, 10, 20, 30, …}.

The set of colors in the rainbow is
{red, orange, yellow, green, blue, indigo, violet}.

As you became bored with simply counting, you learned to add and subtract numbers. This led to a new set of numbers, the **whole numbers**.

> The *whole numbers* are the counting numbers *and* zero {0, 1, 2, 3, …}.

Remember getting negative answers when you added and subtracted? Those **negative numbers** made another set of numbers necessary. The *integers* are the positive numbers, their **opposites** (also called **additive inverses** or *negative numbers*), and zero.

> The integers can be expressed (or written) as {… , -3, -2, -1, 0, 1, 2, 3, …}.
>
> **Even integers** are integers divisible by 2. The integers {… -4, -2, 0, 2, 4 …} form the set of *even integers*.
>
> **Remember:** Every even integer ends with the **digit** 0, 2, 4, 6, or 8 in its ones (or units) place.
>
> **Odd integers** are integers that are *not* divisible by 2. The integers {… -5, -3, -1, 1, 3, 5 …} form the set of *odd integers*.
>
> **Remember:** Every odd integer ends with the *digit* 1, 3, 5, 7, or 9 in its ones place.

Note: There are *no* **fractions** or **decimals** listed in the set of integers above.

When you learned to divide and got answers that were *integers*, *decimals*, or *fractions*, your answers were all from the set of **rational numbers**.

> *Rational numbers* can be expressed as fractions that can then be converted to **terminating decimals** (with a *finite* number of digits) or **repeating decimals** (with an *infinitely* repeating sequence of digits). For example, $-\frac{3}{5} = -0.6$, $\frac{6}{2} = 3$, $-\frac{8}{4} = -2$, and $\frac{1}{3} = 0.333\ldots$ or $0.\overline{3}$.

As you learned more about mathematics, you found that some numbers behaved irrationally. **Irrational numbers** are numbers that cannot be written as a **ratio** of two integers. Their decimals never repeated a **pattern** and never ended.

> *Irrational numbers* like **π (pi)** and $\sqrt{5}$ have non-*terminating*, *non-repeating* **numerals**.

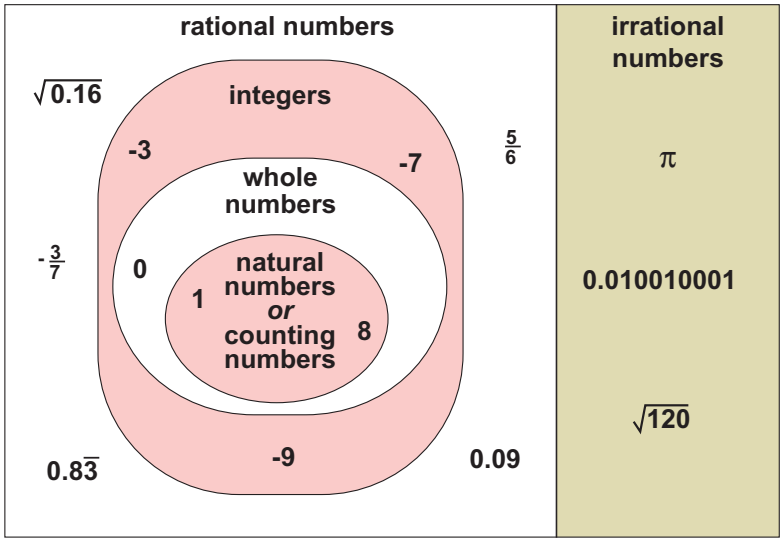
If you put all of the rational numbers and all of the irrational numbers together in a set, you get the set of **real numbers**.

> The set of *real numbers* is often symbolized with a capital R.

A diagram showing the *relationships* among all the sets mentioned is shown below.

Remember: Real numbers include all rational numbers and all irrational numbers.

The Set of Real Numbers



Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. the numbers in the set {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

_____ 2. the numbers in the set {1, 2, 3, 4, 5, …}

_____ 3. a real number that can be expressed as a ratio of two integers

_____ 4. a set in which a whole number can be used to represent its number of elements; a set that has bounds and is limited

_____ 5. the numbers in the set {0, 1, 2, 3, 4, …}

_____ 6. a real number that cannot be expressed as a ratio of two integers

_____ 7. any integer *not* divisible by 2

_____ 8. a set that is *not* finite; a set that has *no* boundaries and *no* limits

_____ 9. any integer divisible by 2

_____ 10. the numbers that result from multiplying a given whole number by the set of whole numbers

_____ 11. the set of all rational and irrational numbers

_____ 12. the symbol designating the ratio of the circumference of a circle to its diameter

A. even integers

B. finite set

C. infinite set

D. integers

E. irrational number

F. multiples

G. natural numbers (counting numbers)

H. odd integers

I. pi (π)

J. rational number

K. real numbers

L. whole number

Practice

Match each **description** *with the correct* **set**. *Write the letter on the line provided.*

_____	1. {2, 3, 4, 5, 6}	A. {counting numbers between 1 and 7}
_____	2. {0, 1, 2, 3}	B. {even integers between -3 and 4}
_____	3. {3, 6, 9, 12, …}	C. {first five counting numbers}
_____	4. {-2, 0, 2}	D. {first four whole numbers}
_____	5. {6, 12, 18, …}	E. {integers that are multiples of 6}
_____	6. {1, 2, 3, 4, 5}	F. {natural-number multiples of 3}
_____	7. {-3, -1, 1, 3}	G. {odd integers between -4 and 5}
_____	8. {… , -18, -12, -6, 0, 6, 12, 18, …}	H. {whole number multiples of 6}

Practice

Write **finite** *if the* **set** *has bounds and is limited. Write* **infinite** *if the* **set** *has* no *boundaries and is* not *limited.*

_____ 1. {whole numbers less than 1,000,000}

_____ 2. {natural numbers with four digits}

_____ 3. {whole numbers with 0 as the last numeral}

_____ 4. {real numbers between 6 and 8}

_____ 5. {counting numbers between 2 and 10}

_____ 6. {first five counting numbers}

_____ 7. {natural-number multiples of 5}

_____ 8. {integers less than 1,000,000}

_____ 9. {counting numbers with three digits}

_____ 10. {whole numbers with 5 as the last numeral}

Practice

Write **True** *if the statement is correct. Write* **False** *if the statement is* not *correct.*

_____ 1. 7 is a rational number.

_____ 2. $\frac{5}{3}$ is a real number.

_____ 3. -9 is a whole number.

_____ 4. 0 is a counting number.

_____ 5. $\sqrt{4}$ is irrational.

_____ 6. $\sqrt{7}$ is a rational number.

_____ 7. $\frac{10}{3}$ is a whole number.

_____ 8. -9 is a natural number.

_____ 9. 0 is an even integer.

_____ 10. π is a real number.

Practice

Use the list below to write the correct term for each definition on the line provided.

additive inverses	**null set (ø)** or **empty set**
element or **member**	**positive numbers**
grouping symbol	**repeating decimal**
negative numbers	**terminating decimal**

_____ 1. a set with no elements or members

_____ 2. parentheses ( ), braces { }, brackets [ ], and fraction bars indicating grouping of terms in an expression

_____ 3. a decimal that contains a finite (limited) number of digits

_____ 4. a decimal in which one digit or a series of digits repeat endlessly

_____ 5. a number and its opposite whose sum is zero (0)

_____ 6. numbers less than zero

_____ 7. numbers greater than zero

_____ 8. one of the objects in a set

Lesson Two Purpose

- Understand concrete and symbolic representations of real *and complex* numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)

The Order of Operations

Algebra can be thought of as a game. When you know the rules, you have a much better chance of winning! In addition to knowing how to add, subtract, multiply, and divide integers, fractions, and decimals, you must also use the **order of operations** correctly.

Although you have previously studied the rules for *order of operations*, here is a quick review.

Rules for Order of Operations

Always start on the *left* and move *to* the *right*.

1.	Do operations inside *grouping symbols* first.	( ) , [ ], **or** $\frac{x}{y}$
2.	Then do all *powers* (exponents) **or** *roots*.	x^2 **or** $\sqrt{x}$
3.	Next do *multiplication* **or** *division*— as they occur from left to right.	x **or** ÷
4.	Finally, do *addition* **or** *subtraction*— as they occur from left to right.	+ **or** –

Remember: The fraction bar is considered a grouping symbol.

Example: $\frac{3x^2 + 8}{2} = (3x^2 + 8) \div 2$

Note: In an **expression** where more than one set of grouping symbols occurs, work within the innermost set of symbols first, then work your way outward.

The order of operations makes sure everyone doing the problem correctly will get the same answer.

Some people remember these rules by using this mnemonic device to help their memory.

Please **P**ardon **M**y **D**ear **A**unt **S**ally*

Please ..... **P**arentheses (grouping symbols)
Pardon ..... **P**owers
My **D**ear ..... **M**ultiplication or **D**ivision
Aunt **S**ally ..... **A**ddition or **S**ubtraction

*Also known as **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally—**P**arentheses, **E**xponents, **M**ultiplication or **D**ivision, **A**ddition or **S**ubtraction.

Remember: You do multiplication **or** division—*as they occur* from *left to right*, and then addition **or** subtraction—*as they occur* from *left to right*.

Study the following.

$$25 - 3 \bullet 2 =$$

There are no grouping symbols. There are no **powers (exponents)** or **roots**. We look for multiplication or division and find multiplication. We multiply. We look for addition or subtraction and find subtraction. We subtract.

$$\begin{aligned} 25 - 3 \bullet 2 &= \\ 25 - 6 &= \\ 19 & \end{aligned}$$

Study the following.

$$12 \div 3 + 6 \div 2 =$$

There are no grouping symbols. There are no *powers* or *roots*. We look for multiplication or division and find division. We divide. We look for addition or subtraction and find addition. We add.

$$\begin{aligned} 12 \div 3 + 6 \div 2 &= \\ 4 + 3 &= \\ 7 & \end{aligned}$$

If the rules were ignored, one might divide 12 by 3 and get 4, then add 4 and 6 to get 10, then divide 10 by 2 to get 5. Agreement is needed—using the agreed-upon *order of operations*.

Study the following.

$$30 - 3^3 =$$

There are no grouping symbols. We look for powers and roots and find powers, 3^3. We calculate this. We look for multiplication or division and find none. We look for addition or subtraction and find subtraction. We subtract.

$$\begin{aligned} 30 - 3^3 &= \\ 30 - 27 &= \\ 3 & \end{aligned}$$

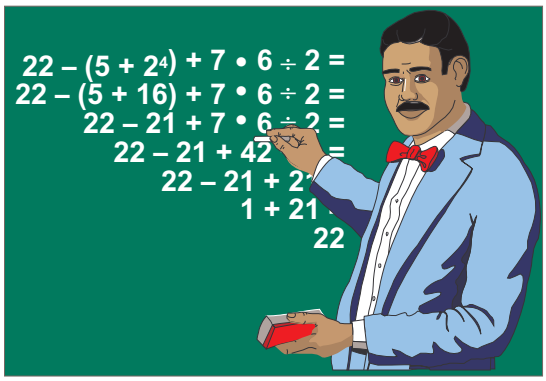
Study the following.

$$22 - (5 + 2^4) + 7 \bullet 6 \div 2 =$$

Please	**P**arentheses
Pardon	**P**owers
My	**M**ultiplication *or*
Dear	**D**ivision
Aunt	**A**ddition *or*
Sally	**S**ubtraction

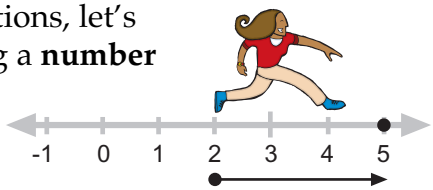
We look for grouping symbols and see them. We must do what is inside the parentheses first. We find addition and a power. We do the power first and then the addition. There are no roots. We look for multiplication or division and find both. We do them in the order they occur, left to right, so the multiplication occurs first. We look for addition or subtraction and find both. We do them in the order they occur, left to right, so the subtraction occurs first.

$$\begin{aligned} 22 - (5 + 2^4) + 7 \bullet 6 \div 2 &= \\ 22 - (5 + 16) + 7 \bullet 6 \div 2 &= \\ 22 - 21 + 7 \bullet 6 \div 2 &= \\ 22 - 21 + 42 \div 2 &= \\ 22 - 21 + 21 &= \\ 1 + 21 &= \\ 22 & \end{aligned}$$



Adding Numbers by Using a Number Line

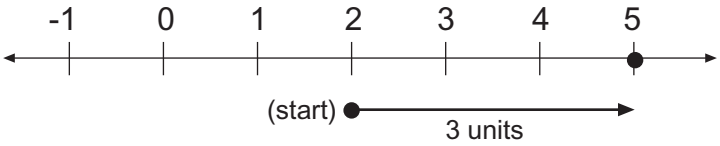
After reviewing the rules for order of operations, let's get a visual feel for adding integers by using a **number line**.



Example 1

Add 2 + 3

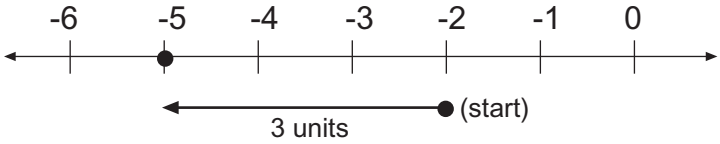
1. Start at 2.
2. Move 3 units to the right in the *positive* direction.
3. Finish at 5.
   So, 2 + 3 = 5.



Example 2

Add -2 + -3

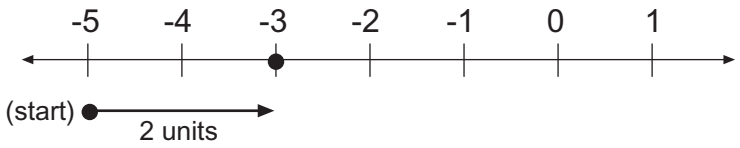
1. Start at -2.
2. Move 3 units to the left in the *negative* direction.
3. Finish at -5.
   So, -2 + -3 = -5.



Example 3

Add -5 + 2

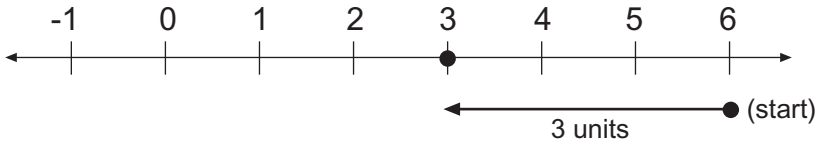
1. Start at -5.
2. Move 2 units to the right in a *positive* direction.
3. Finish at -3.
   So, -5 + 2 = -3.



Example 4

Add 6 + -3

1. Start at 6.
2. Move 3 units to the left in a *negative* direction.
3. Finish at 3.
   So 6 + -3 = 3.



Addition Table

Look for *patterns* in the Addition Table below.

Addition Table

+	4	3	2	1	0	-1	-2	-3	-4
4	8	7	6	5	4	3	2	1	0
3	7	6	5	4	3	2	1	0	-1
2	6	5	4	3	2	1	0	-1	-2
1	5	4	3	2	1	0	-1	-2	-3
0	4	3	2	1	0	-1	-2	-3	-4
-1	3	2	1	0	-1	-2	-3	-4	-5
-2	2	1	0	-1	-2	-3	-4	-5	-6
-3	1	0	-1	-2	-3	-4	-5	-6	-7
-4	0	-1	-2	-3	-4	-5	-6	-7	-8

addends (any numbers being added)

sums (the result of adding numbers together)

addends

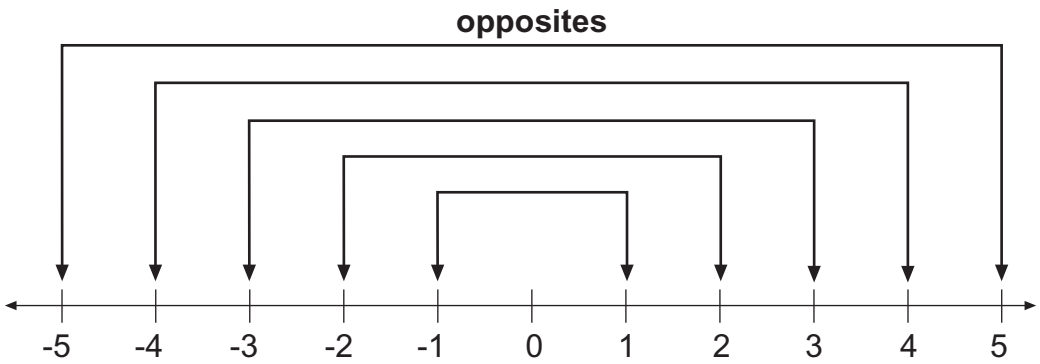
- Look at the *positive sums* in the table. Note the *addends* that result in a positive sum.
- Look at the *negative sums* in the table. Note the *addends* that result in a negative sum.
- Look at the sums that are *zero*. Note the *addends* that result in a sum of zero.
- **Additive Identity Property**—when zero is added to any number, the sum is the number. Note that this property is true for addition of integers.
- **Commutative Property of Addition**—the order in which numbers are added does *not* change the sum. Note that this property is true for addition of integers.
- **Associative Property of Addition**—the way numbers are grouped when added does *not* change the sum. Note that this property is true for addition of integers.

Opposites and Absolute Value

Although we can visualize the process of adding by using a number line, there are faster ways to add. To accomplish this, we must know two things: *opposites* or *additive inverses* and **absolute value**.

Opposites or Additives Inverses

5 and -5 are called *opposites*. Opposites are two numbers whose points on the number line are the same distance from 0 but in opposite directions.



The opposite of 0 is 0.

Every **positive integer** can be paired with a **negative integer**. These pairs are called *opposites*. For example, the opposite of 4 is -4 and the opposite of -5 is 5.

The opposite of 4 can be written -(4), so -(4) equals -4.

$$-(4) = -4$$

The opposite of -5 can be written -(-5), so -(-5) equals 5.

$$-(-5) = 5$$

Two numbers are opposites or *additive inverses* of each other if their sum is zero.

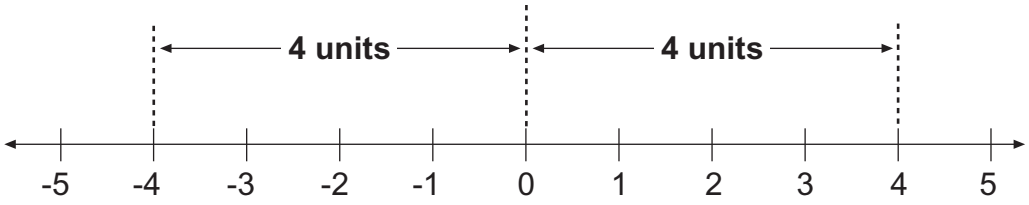
For example: $4 + -4 = 0$
$-5 + 5 = 0$

Absolute Value

The *absolute value* of a number is the distance the number is from the *origin* or zero (0) on a number line. The symbol | | placed on either side of a number is used to show absolute value.

Look at the number line below. -4 and 4 are different numbers. However, they are the same distance in number of units from 0. Both have the same *absolute value* of 4. Absolute value is *always* positive because distance is always positive—you cannot go a negative distance. The absolute value of a number tells the number's *distance* from 0, not its *direction*.

$|-4| = |4| = 4$ The absolute value of a number is *always* positive.



The absolute value 0 is 0.

$|-4|$ denotes the absolute value of -4.

$|-4| = 4$

$|4|$ denotes the absolute value of 4.

$|4| = 4$

The absolute value of 10 is 10. We can use this notation:

$| 10 | = 10$

The absolute value of -10 is also 10. We can use this notation:

$| -10 | = 10$

Both 10 and -10 are 10 units away from the origin. Consequently, the absolute value of both numbers is 10.

The absolute value of 0 is 0.

$| 0 | = 0$

The *opposite* of the absolute value of a number is *negative*.

$-| 8 | = -8$

Now that we have this terminology under our belt, we can introduce two rules for adding numbers which will enable us to add quickly.

Adding Positive and Negative Integers

There are specific rules for adding positive and negative numbers.

1. **If the two integers have the *same sign*, *keep* the *sign* and *add* their absolute values.**

   Example
   -5 + -7

   Think: Both signs are negative.

   $| -5 | = 5$

   $| -7 | = 7$

   $5 + 7 = 12$

   The sign will be negative because both signs were negative. Therefore, the answer is -12.

   $-5 + -7 = -12$

2. **If the two integers have *opposite signs*, *subtract* the absolute values. The answer has the *sign* of the integer with the *greater* absolute value.**

Example
-8 + 3

Think: Signs are opposite.

| -8 | = 8

| 3 | = 3

8 – 3 = 5

The sign will be negative because 8 has the greater absolute value. Therefore, the answer is -5.

-8 + 3 = -5

Example
-6 + 8

Think: Signs are opposite.

| -6 | = 6

| 8 | = 8

8 – 6 = 2

The sign will be positive because 8 has a greater absolute value. Therefore, the answer is 2.

-6 + 8 = 2

Example
5 + -7

Think: Signs are opposite.

| 5 | = 5

| -7 | = 7

7 – 5 = 2

The sign will be negative because 7 has the greater absolute value. Therefore, the answer is -2.

5 + -7 = -2

Rules to Add Integers

- The sum of two positive integers is *positive*. (+) + (+) = +
- The sum of two negative integers is *negative*. (-) + (-) = -
- The sum of a positive integer and a negative integer takes the *sign of the greater absolute value*. (-) + (+) = / (+) + (-) = } use sign of number with greatest absolute value
- The sum of a positive integer and a negative integer is zero if numbers have the *same absolute value*. (*a*) + (-*a*) = 0 / (-*a*) + (*a*) = 0

Check Yourself Using a Calculator When Adding Positive and Negative Integers

Use a **calculator** with a [+/-] **sign-change** key.

For example, for -16 + 4, you would enter 16 [+/-] [+] 4 [=] and get the answer -12.

Subtracting Integers

In the last section, we saw that 8 plus -3 equals 5.

$$8 + (-3) = 5$$

From elementary school we know that 8 minus 3 equals 5.

$$8 - 3 = 5$$

Below are similar examples.

$10 + (-7) = 3$	$12 + (-4) = 8$
$10 - 7 = 3$	$12 - 4 = 8$

These three examples show that there is a connection between adding and subtracting. As a matter of fact we can make any subtraction problem into an addition problem and vice versa.

This idea leads us to the following definition:

Definition of Subtraction
$$a - b = a + (-b)$$

Examples

$8 - 10 = 8 + (-10) = -2$

$12 - 20 = 12 + (-20) = -8$

$-2 - 3 = -2 + (-3) = -5$

Even if we have $8 - (-8)$, this becomes

8 plus the opposite of -8, which equals 8.

$8 + -(-8) =$

$8 + 8 = 16$

And -9 – (-3), this becomes

-9 plus the opposite of -3, which equals 3.

-9 + -(-3)

-9 + 3 = -6

Shortcut Two negatives become one positive!

10 – (-3) becomes 10 plus 3.

10 + 3 = 13

And -10 – (-3) becomes -10 plus 3.

-10 + 3 = -7

Generalization: Subtracting Integers
Subtracting an integer is the same as adding its opposite. $a - b = a + (-b)$

Check Yourself Using a Calculator When Subtracting Negative Integers

Use a **calculator** with a [+/-] **sign-change** key.

For example, for 18 – (-32), you would enter
18 [–] 32 [+/-] [=] and get the answer 52.

Practice

Simplify *the following* **expressions**. *Show* **essential** *steps.*

Example: $5 - (8 + 3) =$
$5 - 11 =$
-6

1. $9 - (5 - 2 + 6) =$

2. $(7 - 3) + (-5 + 3) =$

3. $(5 + 32 - 36) + (12 + 5 - 10) =$

4. $(-26 + 15 - 13) - (4 - 16 + 43) =$

5. $(-15 + 3 - 7) - (26 - 14 + 10) =$

✓ **Check yourself:** The *sum* of the correct answers from numbers 1-5 above is -86.

Multiplying Integers

What *patterns* do you notice?

$3(4) = 12$	$3(-4) = -12$
$2 \bullet 4 = 8$	$2 \bullet -4 = -8$
$1(4) = 4$	$1(-4) = -4$
$0 \bullet 4 = 0$	$0 \bullet -4 = 0$
$-1(4) = -4$	$-1(-4) = 4$
$-2 \bullet 4 = -8$	$-2 \bullet -4 = 8$
$-3(4) = -12$	$-3(-4) = 12$

Ask yourself:

- What is the sign of the **product** of two positive integers?
  $3(4) = 12$ $2 \bullet 4 = 8$ *positive*
- What is the sign of the *product* of two negative integers?
  $-1(-4) = 4$ $-2 \bullet -4 = 8$ *positive*
- What is the sign of the product of a positive integer and a negative integer or a negative integer and a positive integer?
  $3(-4) = -12$ $-2 \bullet 4 = -8$ *negative*
- What is the sign of the product of any integer and 0?
  $0 \bullet 4 = 0$ $0 \bullet -4 = 0$ *neither*; zero is neither positive or negative

You can see that the sign of a *product* depends on the signs of the numbers being multiplied. Therefore, you can use the following rules to multiply integers.

Rules to Multiply Integers

• The product of two positive integers is *positive*.	(+)(+) = +
• The product of two negative integers is *positive*.	(-)(-) = +
• The product of two integers with different signs is *negative*.	(+)(-) = - (-)(+) = -
• The product of any integer and 0 is *0*.	(*a*)(0) = 0 (-*a*)(0) = 0

Check Yourself Using a Calculator When Multiplying Integers

Use a **calculator** with a [+/-] **sign-change** key.

For example, for -13 • -7, you would enter
13 [+/-] [x] 7 [+/-] [=] and get the answer 91.

Dividing Integers

Think:

1. What would you multiply $^{+}6$ by to get $^{+}42$?
   $^{+}6 \bullet ? = {}^{+}42$
   Answer: $^{+}7$ because $^{+}6 \bullet {}^{+}7 = {}^{+}42$

2. What would you multiply -6 by to get -54?
   $-6 \bullet ? = -54$
   Answer: $^{+}9$ because $-6 \bullet {}^{+}9 = -54$

3. What would you multiply -15 by to get 0?
   $-15 \bullet ? = 0$
   Answer: 0 because $-15 \bullet 0 = 0$

Remember: The result of dividing is a **quotient**.

Example

42 divided by 7 results in a *quotient* of 6.

$$42 \div 7 = \underset{\text{quotient}}{6}$$

To find the quotient of 12 and 4 we write:

$$4\overline{)12} \quad \textit{or} \quad 12 \div 4 \quad \textit{or} \quad \frac{12}{4}$$

Each problem above is read "12 *divided by* 4." In each form, the quotient is 3.

$$\text{divisor} \rightarrow 4\overline{)\overset{\text{quotient}}{3}12} \quad \textit{or} \quad \underset{\text{dividend}}{12} \overset{\text{divisor}}{\div} 4 = \underset{\text{quotient}}{3} \quad \textit{or} \quad \frac{\overset{\text{dividend}}{12}}{\underset{\text{divisor}}{4}} = 3 \leftarrow \text{quotient}$$

In $\frac{12}{4}$, the bar separating 12 and 4 is called a *fraction bar*. Just as *subtraction* is the *inverse of addition*, *division* is the *inverse of multiplication*. This means that *division* can be *checked by multiplication*.

$4\overline{)12}$ = 3 *because* $3 \bullet 4 = 12$

Division of integers is *related to* multiplication of integers. The sign rules for division can be discovered by writing a related multiplication problem.

For example,

same signs → positive quotient (or product)	**different signs → negative quotient (or product)**
$\frac{6}{2} = 3$ because $3 \bullet 2 = 6$	$\frac{-6}{2} = -3$ because $-3 \bullet 2 = -6$
$\frac{-6}{-2} = 3$ because $3 \bullet -2 = -6$	$\frac{6}{-2} = -3$ because $-3 \bullet -2 = 6$

Below are the rules used to divide integers.

Rules to Divide Integers

- The quotient of two positive integers is *positive*. $(+) \div (+) = +$
- The quotient of two negative integers is *positive*. $(-) \div (-) = +$
- The quotient of two integers with different signs is *negative*. $(+) \div (-) = -$ $(-) \div (+) = -$
- The quotient of 0 divided by any nonzero integer is *0*. $0 \div a = 0$

Note the special division properties of 0:

$0 \div 9 = 0$	$0 \div -9 = 0$
$\frac{0}{5} = 0$	$\frac{0}{-5} = 0$
$15\overline{)0}$ = 0	$-15\overline{)0}$ = 0

Remember: Division by 0 is *undefined*. The quotient of any number and 0 is not a number.

We say that $\frac{+9}{0}$, $\frac{+5}{0}$, $\frac{+15}{0}$, $\frac{-9}{0}$, $\frac{-5}{0}$, and $\frac{-15}{0}$ are *undefined*.

Likewise, $\frac{0}{0}$ is undefined.

For example, try to divide 134 ÷ 0. To divide, think of the related multiplication problem.

? x 0 = 134

Any number times 0 is 0—so mathematicians say that division by 0 is undefined.

Check Yourself Using a Calculator When Dividing Integers

Use a **calculator** with a [+/-] **sign-change** key.

For example, for $\frac{-54}{9}$, you would enter
54 [+/-] [÷] 9 [=] and get the answer -6.

Properties of Addition and Multiplication

Let's examine some basic *properties* which will help us work with simple to more complex **equations**.

Order (Commutative Property)	
Commutative Property of Addition: Numbers can be added in any order and the sum will be the same. $10 + 2 = 2 + 10$ $x + 2 = 2 + x$	**Commutative Property of Multiplication:** Numbers can be multiplied in any order and the product will be the same. $2 \bullet 10 = 10 \bullet 2$ $2 \bullet x = x \bullet 2$
Grouping (Associative Property)	
Associative Property of Addition: Numbers can be grouped in any order and the sum will be the same. $(5 + 3) + 2 = 5 + (3 + 2)$ $(5 + x) + y = 5 + (x + y)$	**Associative Property of Multiplication:** Numbers can be grouped in any order and the product will be the same. $(5 \bullet 3) \bullet 2 = 5 \bullet (3 \bullet 2)$ $(5 \bullet x) \bullet y = 5 \bullet (x \bullet y)$
Identity Properties	
Additive Identity: The sum of any number and zero is the number. $5 + 0 = 5$ $x + 0 = x$	**Multiplicative Identity:** The product of any number and one is the number. $5 \bullet 1 = 5$ $x \bullet 1 = x$
Inverse Properties	
Additive Inverse: The sum of any number and its additive inverse is 0. $3 + -3 = 0$ 3 and -3 are additive inverses, also called *opposites*.	**Multiplicative Inverse:** The product of any number and its multiplicative inverse (**reciprocal**) is 1. $4 \times \frac{1}{4} = 1$ 4 and $\frac{1}{4}$ are multiplicative inverses, also called *reciprocals*.

Practice

Simplify *the following. Show* **essential steps**.

Example: $\frac{2(-3 \bullet 6)}{-4} =$

$\frac{2(-18)}{-4} =$

$\frac{-36}{-4} =$

9

1. $\frac{(6)(-5)(3)}{9} =$

2. $(-3)(5)(\frac{4}{3})(-2) =$

3. $(\frac{1}{2})(-4)(0)(5) =$

4. $\frac{-3(4)(-2)(5)}{(-16)} =$

5. $\dfrac{6(\frac{4}{7})(-\frac{3}{2})(-2)}{-(\frac{3}{7})} =$

6. $\left[\dfrac{7-(-3)}{5-3}\right]\left[\dfrac{4+(-8)}{3-5}\right] =$

7. $\left[\dfrac{12+(-2)}{3+(-8)}\right]\left[\dfrac{6+(-15)}{8-5}\right] =$

8. $\left\{7+3\left[\dfrac{6+(-18)}{4+2}\right]-5\right\}+5 =$

9. $\left\{4-2\left[\dfrac{5-(-4)}{2+1}-\dfrac{6}{3}\right]\right\}+1 =$

10. $\dfrac{3(3+2)-3\bullet 3+2}{3\bullet 2+2(2-1)} =$

Practice

Use the given **value** *of each* **variable** *to* **evaluate each expression**. *Show* **essential steps**.

Example: Evaluate $5\left(\frac{F-32}{9}\right) =$

$\boxed{F = 212}$

Replace F with 212 and simplify.

$$5\left(\frac{212-32}{9}\right) =$$

$$5\left(\frac{180}{9}\right) =$$

$$5(20) =$$

$$100$$

1. $\boxed{E = 18 \quad e = 2 \quad R = 6}$

   $\frac{E-e}{R} =$

2. $\boxed{P = 1{,}000 \quad r = 0.04 \quad t = 5}$

   $P + Prt =$

3. $\boxed{r = 8 \quad h = 6}$

   $2r(r + h) =$

Practice

Simplify *the following. Show* **essential steps**.

Example: $(4 + 1)^2 - \frac{4 \bullet 3^2}{6} =$

$$5^2 - \frac{4 \bullet 9}{6} =$$

$$25 - \frac{36}{6} =$$

$$25 - 6 =$$

$$19$$

1. $\frac{8 \bullet 2^2}{4^2} + (3 \bullet 1)^2 =$

2. $\frac{5^2 \bullet 3^2}{4} - (2 + 1)^2 =$

3. $\frac{3^2 \bullet 2^2}{7 - 2^2} + \frac{(-3)(2)^2}{6 - 3} =$

4. $\dfrac{7^2 - 6^2}{10 + 3} + \dfrac{8^2 \bullet (-2)}{(-2)^4} =$

5. $\dfrac{(-5)^2 - 3^2}{4 - 6} + \dfrac{-(3)^2 \bullet 2}{5 + 1} =$

Use the given **value** *of each* **variable** *to* **evaluate the following expressions**. *Show* **essential steps**.

$x = 3$	$y = -2$

6. $\dfrac{-xy^2}{6} + 2x^2y =$

7. $(x + y)^2 + (x - y)^2 =$

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. the order in which two numbers are added or multiplied does *not* change their sum or product, respectively

_____ 2. the order of performing computations in parentheses first, then exponents or powers, followed by multiplication and/or division (as read from left to right), then addition and/or subtraction (as read from left to right)

_____ 3. a number's distance from zero (0) on a number line; distance expressed as a positive value

_____ 4. the number one (1); the product of a number and the multiplicative identity is the number itself

_____ 5. two numbers whose product is 1

_____ 6. the number of times the base occurs as a factor

_____ 7. any symbol, usually a letter, which could represent a number

_____ 8. the way in which three or more numbers are grouped for addition or multiplication does *not* change their sum or product, respectively

_____ 9. any two numbers with a product of 1

_____ 10. the number zero (0); when zero (0) is added to another number the sum is the number itself

A. absolute value

B. additive identity

C. associative property

D. commutative property

E. exponent

F. multiplicative identity

G. multiplicative inverse

H. order of operations

I. reciprocals

J. variable

Lesson Three Purpose

- Understand concrete and symbolic representations of real *and complex* numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)

Percent

A **percent (%)** is a *ratio* because it compares a number to 100.

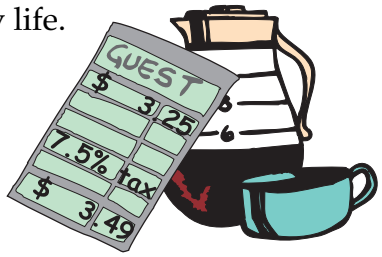
A *percent* is a special fraction with a **denominator** of 100.

$$26\% = \frac{26}{100} \qquad 43\% = \frac{43}{100} \qquad 6\% = \frac{6}{100} \qquad 125\% = \frac{125}{100}$$

Remember that fractions are sometimes written as decimals.

$$0.38 = \frac{38}{100} = 38\% \qquad 0.07 = \frac{7}{100} = 7\% \qquad 0.61 = \frac{61}{100} = 61\% \qquad 4.87 = \frac{487}{100} = 487\%$$

We often need to figure percentages in our daily life. Sales tax is a percentage added to purchases. Tipping is usually based on a percentage of the bill as well. When items are on sale, they are usually advertised at a discount, which is a percentage of the original price.



Sales tax is a percentage added to purchases.

Sales Tax

If Sasha bought $54.34 worth of school supplies and the sales tax rate is 8%, how much will Sasha owe?

Here are two methods to figure Sasha's total.

Method One

- One way is to multiply $54.34 by 0.08, which is the decimal **equivalent** of 8%. This will give you the amount of tax she owes. Add the tax to the purchase price to find Sasha's total.

$54.34 x 0.08 = 4.3472 ← Because we are working with money, we must **round** *up* to the *nearest cent*. So the tax owed is $4.35.

change to a 5

$54.34 + $4.35 = $58.69 ← purchase price + tax = total cost

Method Two

- Another way to **solve** for Sasha's total is to multiply her purchase price by 100% and 8% all at the same time. 108% represents the price she'll pay for the supplies (100%), plus the tax (8%). With the use of a calculator, this can be done in one step.

Remember: Change 108% to the decimal *equivalent*, 1.08. To change *percent of a number* to a *decimal number*, do the following:

108% = 1.08

move decimal two places to the left

$54.34 x 1.08 = $58.6872 ← *Round* the money up to the nearest cent.

change to a 9

$58.69 ← Sasha's total cost

Tipping

You can use the same methods for tipping as you use with sales tax. When you intend to give the tip directly to a person, multiply the total by the tax percentage, like step one in Method One on the previous page. When you are adding the tip to your bill to pay in one lump sum, use Method Two on the previous page.

Note: Tips and taxes should *always* be *less than* the original cost!

Discounts and Price Increases

A *discount* is an amount of the original cost (usually a percentage) that you do *not* have to pay. A *price increase* is the *opposite* of a discount. It is also usually a percentage that you must pay *above* the original cost of an item.

Hint: It is important to read a problem very carefully. Sometimes you may need to read it more than once to determine if you are looking for the *amount of a discount*, the *amount of a price increase*, or the *new cost*. Before you finish with a problem, reread it to make sure that you have answered the question presented in the problem.

Example 1

Pete wants to buy a new suit at a store that is advertising a 20% *discount* on all items. The suit Pete likes has a price tag of $154. How much will the suit *cost on sale*?

- **Method One**

  One method you could use is to multiply $154 by 20% and then *subtract* that amount from $154.

  $154 x 0.20 = $30.80 ← This is the amount Pete will save.

  $154 – $30.80 = $123.20 ← Since the problem asked you to find his *cost*, you must now *subtract* the *discount* $30.80 from $154.

- **Method Two**

  Another method uses the fact that Pete will pay 80% of the total cost, because 100% – 20% = 80%.

  $154 x 0.80 = $123.20 ← Multiplying the original cost $154 by 80% gives Pete's cost.

Example 2

Alejandro must increase the prices in his restaurant to keep from losing money. He is planning on a 12% *price increase*. If a hamburger platter costs $4.85, how much will it cost at the new price?

- **Method One**

  $4.85 x 0.12 = 0.582 ← The amount of the *increase* will be $0.59 (round money *up* to the nearest cent).

  $4.85 + $0.59 = $5.44 ← Original price *plus* the amount of the increase.

  OR

- **Method Two**

  $4.85 x 1.12 = $5.44 ← The original price times 112%.

Practice

Answer the following. **Round *up* money to the nearest cent. Show all your work.**

1. Scott wants to leave a 20% tip for his waitress. His bill is $27.52. How much of a *tip* should he give her?

   Answer: _____

2. Jennifer bought 4 new tires for her car. Each tire cost $26.65. If the sales tax rate is 7%, what was her *total cost*?

   Answer: _____

3. Elena expects to get a 15% tip from a customer whose bill is $66.98. If the customer adds the tip to his bill, what will his *total cost* be?

   Answer: _____

4. Franco bought new speakers for his car. They cost $176.54. How much did he have to *pay* if the sales tax was 8%?

   Answer: _____

5. Julia bought a $66 pair of shoes at a 30% discount. How much did Julia *pay for the shoes*?

Answer: _____

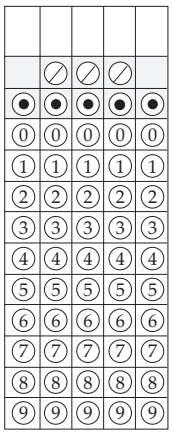
6. Leonard gets an employee discount at the computer store where he works. How much will his 25% discount *save* him on a $699 computer?

Answer: _____

Numbers 7-9 are **gridded-response items**.
Write answers along the top of the grid and correctly mark them below. **Round *up* to the nearest cent. Show all your work.**

7. Sara wants to increase the amount she charges to cut lawns by 35%. If she currently charges $40 to cut a lawn, what will her *new charge* be?

Mark your answer on the grid to the right.



8. Louis took his date to dinner. The bill was $65 without tax. The tax rate in Louis' town is 8%. He wants to leave a tip of 20% based on the *total bill including the tax*. How much will Louis *spend on this date*?

Mark your answer on the grid to the right.

	⊘	⊘	⊘	
●	●	●	●	●
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4
5	5	5	5	5
6	6	6	6	6
7	7	7	7	7
8	8	8	8	8
9	9	9	9	9

9. Cynthia works in a shoe store that is having a 40% off sale. She also gets an employee discount of 15%. She wants to buy a pair of shoes that were originally priced at $56.99. How much will she *save*?
Hint: Add discounts together before multiplying.

Mark your answer on the grid to the right.

	⊘	⊘	⊘	
●	●	●	●	●
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4
5	5	5	5	5
6	6	6	6	6
7	7	7	7	7
8	8	8	8	8
9	9	9	9	9

Practice

Answer the following. **Round *up* money to the nearest cent. Show all your work.**

1. Danielle bought 4 new tires for her truck. Each tire cost $56.65. If the sales tax rate is 9%, how much was the *tax*?

   Answer: _____

2. Elena expects to get a 20% tip from a customer whose bill is $54.89. If the customer adds the 20% tip to his bill, what will his *total cost* be?

   Answer: _____

3. Justin wants to leave a 15% tip for his waiter. His bill is $16.45. How much of a *tip* should he give him?

   Answer: _____

4. Lara wants to *increase* the amount she charges to rake leaves by 25%. If she currently charges $40 to rake a lawn, what will her *increase* be?

   Answer: _____

5. Joseph works in a shoe store that is having a 50% off sale. He also gets an employee discount of 15%. He wants to buy a pair of shoes that were originally priced at $126.99. How much will he *pay*?

   Answer: _____

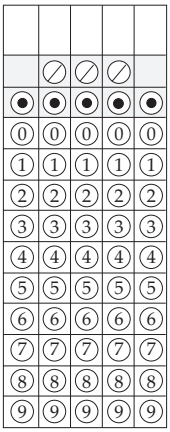
6. Juaquin bought new speakers for his car. They cost $342.54. How much did he have to *pay* if the sales tax was 8%?

   Answer: _____

Number 7 is a **gridded-response item**.
Write answer along the top of the grid and correctly mark it below. **Round *up* to the nearest cent. Show all your work.**

7. Megan bought a $96 bottle of perfume at a 33% discount. How much did Megan *pay* for the perfume?

   Mark your answer on the grid to the right.



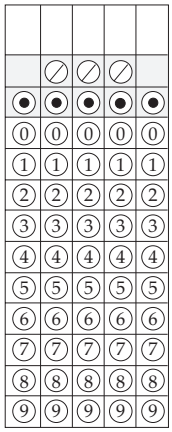
8. Matthew gets an employee discount at the electronics store where he works. How much will his 25% discount *save* him on a $559 CD player?

   Answer: _____

Number 9 is a **gridded-response item**.
Write answer along the top of the grid and correctly mark it below. **Round *up* to the nearest cent. Show all your work.**

9. Freda took her date to dinner. The bill was $75 without tax. The tax rate in Freda's town is 9%. She wants to leave a tip of 20% based on the *total bill plus tax*. How much will Freda *spend on this date*?

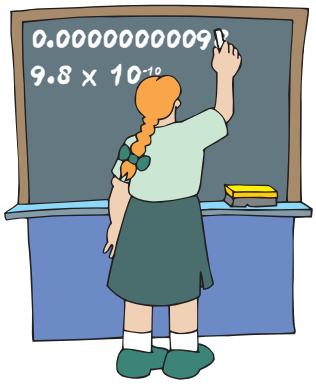
   Mark your answer on the grid to the right.



Lesson Four Purpose

- Understand concrete and symbolic representations of real *and complex* numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)

Scientific Notation



Scientists often use a shorthand method of writing very large or very small numbers.

Scientists often use a shorthand method of writing very large or very small numbers. They use *exponents* to express a number as a product of a power of 10 and a decimal number that is *greater than* or *equal to* (≥) 1 but *less than* 10. This is called **scientific notation** and is based on the concept that it is easier to read exponents than it is to count zeroes.

Here are some examples.

5.68 is a decimal ≥ 1 and < 10.

Power of 10 with positive exponent 5.

Count the number of places needed to move the decimal point to get a number between 1 and 10.

$$568{,}000 = 5.68 \times 10^{5}$$

Notice that the exponent 5 indicates that the decimal was moved five places to the left.

Write the number as a product of a number between 1 and 10 and a power of 10.

3.25 is a decimal ≥ 1 and < 10.

Power of 10 with negative exponent -6.

Count the number of places needed to move the decimal point to get a number between 1 and 10.

$$0.00000325 = 3.25 \times 10^{-6}$$

Notice that the exponent is negative because we had to move the decimal six places to the right.

Write the number as a product of a number between 1 and 10 and a power of 10.

Consider these large quantities:	In scientific notation:
700,000,000	7×10^{8}
980,000	9.8×10^{5}
40,000,000	4×10^{7}
250,000,000,000	2.5×10^{11}

And these very small quantities:	In scientific notation:
0.0085	8.5×10^{-3}
0.000009	9×10^{-6}
0.000000556	5.56×10^{-7}
0.0000302	3.02×10^{-5}

Practice

Write the following in **scientific notation**. *Use* **exponents** *to express the number as a* **power** *of 10 and a* **decimal number** *≥ 1 and < 10.*

1. 36,000 _____
2. 13,200,000 _____
3. 120,000 _____
4. 0.0000053 _____
5. 0.0029 _____

Write the following without *using* **exponents**.

6. 3.45×10^{5} _____
7. 6.754×10^{8} _____
8. 6.34×10^{-4} _____
9. 5.98×10^{6} _____
10. 1.23×10^{-3} _____

Practice

Write the following in **scientific notation**. *Use* **exponents** *to express the number as a* **power** *of 10 and a* **decimal number** *≥ 1 and < 10.*

1. 54,200 _____
2. 0.00034 _____
3. 0.0056 _____
4. 65,000 _____
5. 0.00547 _____

Write the following without *using* **exponents**.

6. 4.87×10^{-8} _____
7. 6.03×10^{-2} _____
8. 5.45×10^{5} _____
9. 6.754×10^{4} _____
10. 3.34×10^{-8} _____

Lesson Five Purpose

- Understand concrete and symbolic representations of real *and complex* numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)

Working with Absolute Value

As discussed earlier in this unit, the *absolute value* of a number is actually the distance that number is from zero on a number line. Because distance is *always* positive, the result when taking the absolute value of a number is *always* positive.

The symbols for absolute value | | can also act as *grouping symbols*. Perform any operations within the grouping symbols first, just as you would within parentheses.

Look at these examples. Notice the similarities and differences in each pair.

$|-7| + |5| = 7 + 5 = 12$ $\qquad$ $|-7 + 5| = |-2| = 2$

$|6| - |-10| = 6 - 10 = -4$ $\qquad$ $|6 - -10| = |6 + 10| = |16| = 16$

Practice

Answer the following. **Perform any operations *within* the grouping symbols first.**

1. $| -23 + 37 | =$

2. $| 21 - 44 | =$

3. $| 16 + 4 | - | 32 | =$

4. $| 16 + 4 | - | -32 | =$

5. $22 - | -10 | + | 56 | =$

✓ **Check yourself:** The *sum* of the answers from numbers 1-5 is $| -81 |$.

Practice

Use the given **value** *for each* **variable** *to* **evaluate the following expressions**. **Perform any operations *within* the grouping symbols first.**

$a = -5$	$b = 7$	$c = -9$

1. $|a| + |b| - |c| =$

2. $|a + b| - |c| =$

3. $|c - a| - |b| =$

4. $|b + c| + |a| =$

5. $|c - b| + |a| =$

Use the given **value** *for each* **variable** *to* **evaluate the following expressions**. **Perform any operations *within* the grouping symbols first.**

$a = -5$	$b = 7$	$c = -9$

6. $|a + c| - |-c| =$

7. $|a + b + c| - |c - b| =$

8. $|a| + |b| + |c| =$

9. $a - |b| - |c| =$

10. $a + |-b| - |c| =$

Practice

Use the given **value** *for each* **variable** *to* **evaluate the following expressions**. **Perform any operations *within* the grouping symbols first.**

$a = 6$	$b = -7$	$c = -8$

1. $|a| + |b| - |c| =$

2. $|a + b| - |c| =$

3. $|c - a| - |b| =$

4. $|a + b + c| - |c - b| =$

5. $|a| + |b| + |c| =$

6. $a - |b| - |c| =$

7. $a + |-b| - |c| =$

Answer the following. **Perform any operations *within* the grouping symbols first.**

8. $| -33 + 57 | =$

9. $| 16 - 34 | =$

10. $| 26 + 4 | - | 36 | =$

11. $| 26 + 4 | - | -36 | =$

12. $22 - | 20 | + | 32 | =$

Practice

Use the list below to complete the following statements.

element or **member**	**irrational**	**rational**
even	**finite**	**real numbers**
exponent	**odd**	**scientific notation**
grouping symbols	**percent (%)**	**variable**

1. Scientists often use _____ to represent very large or very small numbers.

2. A special-case ratio which compares numbers to 100 is called a(n) _____ .

3. The color green is a(n) _____ of the set of colors in the rainbow.

4. A(n) _____ number *cannot* be expressed as a ratio of two integers.

5. { } and [ ] are examples of _____ .

6. Rational numbers and irrational numbers together make up the set of _____ .

7. Any symbol, usually a letter, which could represent a number in a mathematical expression is a(n) _____ .

8. A(n) _____ number can be expressed as a ratio of two integers.

9. Any integer *not* divisible by 2 is called a(n) _____ integer.

10. Any integer divisible by 2 is called a(n) _____ integer.

11. A set in which a whole number can be used to represent its number of elements is a(n) _____ set.

12. In scientific notation, a(n) _____ is used to express a number as a product of 10.

Unit Review

Specify the following sets by **listing the elements** *of each.*

1. {the whole numbers less than 8} _____

2. {the odd counting numbers less than 12} _____

3. {even integers between -5 and 6} _____

Write **finite** *if the set has bounds and is limited. Write* **infinite** *if the set has* no *boundaries and is* not *limited.*

_____ 4. {the colors in a crayon box}

_____ 5. {rational numbers}

_____ 6. {negative integers}

Write **True** *if the statement is correct. Write* **False** *if the statement is* not *correct.*

_____ 7. π is rational.

_____ 8. 0 is a whole number.

_____ 9. -9 is a counting number.

Simplify the following. Show **essential steps**.

Remember: Order of operations—**P**lease **P**ardon **M**y **D**ear **A**unt **S**ally. (Also known as **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally.)

10. $\dfrac{(5)(-2)(7)}{10} =$

11. $\dfrac{(-6)(4) - (8)(2)}{9 - 4} =$

12. $\left[\dfrac{16 - (-4)}{10 - 6}\right]\left[\dfrac{19 + (-8)}{(-2)(3)}\right] =$

13. $\left\{6 - 3\left[\dfrac{5 - (-4)}{-3}\right]\right\} + 5 =$

Use the given **value** *of each* **variable** *to* **evaluate each expression**. *Show* **essential steps**.

14. | $P = 100$ | $r = 0.02$ | $t = 6$ |

$Prt =$

15. | $r = 6$ | $h = 8$ |

$2r(r + h) =$

16. | $x = -2$ | $y = 3$ |

$\frac{-xy^2}{6} + 2xy^2 =$

Simplify the following. Show **essential steps**.

17. $\frac{5^2 + (2^2 - 1)^3}{3^2 - 5} =$

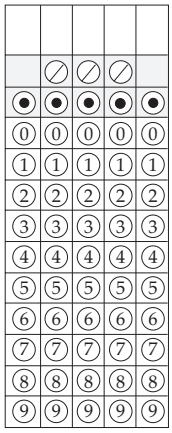
18. $\frac{3^2 \bullet 2(4)}{3^2 - 2^2 + 1} + \frac{5^2 + 7}{2^3} =$

Number 19 is a **gridded-response item**.
Write answer along the top of the grid and correctly mark it below.

Solve the following. **Round *up* to the nearest cent. Show all your work.**

19. Nan wants to tip her waitress 20%. If Nan's bill is $26.32, how much will Nan leave as a *tip*?

    Mark your answer on the grid to the right.



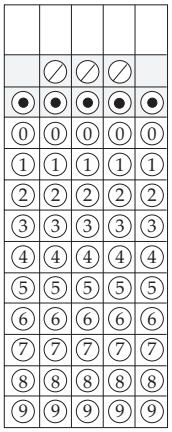
20. Felix gets an 18% discount as an employee of an electronics store. He wants to buy a $198 stereo. How much will he *pay for the stereo*?

    Answer: _____

Number 21 is a **gridded-response item**.
Write answer along the top of the grid and correctly mark it below.

21. Jon bought a new shirt and a pair of slacks for a total of $76.38. The sales tax in his city is 7.5%. Including sales tax, what is Jon's *total cost*? (**Note:** 7.5% = 0.075)

    Mark your answer on the grid to the right.



Write the following in **scientific notation**. *Use* **exponents** *to express the number as a* **power** *of 10 and a* **decimal number** *≥ 1 and < 10.*

22. 385,000 _____

23. 0.0046 _____

Write the following without *using* **exponents**.

24. 2.63×10^4 _____

25. 2.54×10^{-3} _____

Answer the following. **Perform any operations *within* the grouping symbols first.**

26. $|13 - 24| =$

27. $|-19 + 17| - |41 + 8| =$

28. $36 - |14 - 10| + |3 - 15| =$

Use the given **value** *for each* **variable** *to* **evaluate the following** *expressions.* **Perform any operations *within* the grouping symbols first. Show essential steps.**

$a = -6$	$b = -2$	$c = 4$

29. $|a + b| - |c - a| =$

30. $|b + c| + |-a - b| =$

Unit 2: Working with Polynomials

This unit emphasizes the strategies necessary for operations involving polynomials.

Unit Focus

Number Sense, Concepts, and Operations

- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil and calculator. (MA.A.3.4.3)

Algebraic Thinking

- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)

Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

additive inverses ..... a number and its opposite whose sum is zero (0); also called *opposites*
Example: In the equation 3 + -3 = 0, 3 and -3 are additive inverses, or *opposites*, of each other.

base (of an exponent) (algebraic) ..... the number used as a factor in exponential form
Example: 2^3 is the exponential form of 2 x 2 x 2. The numeral two (2) is called the base, and the numeral three (3) is called the exponent.

binomial ..... the sum of two monomials; a polynomial with exactly *two* terms
Examples: $4x^2 + x$ $\quad 2a - 3b$ $\quad 8qrs + qr^2$

canceling ..... dividing a numerator and a denominator by a common factor to write a fraction in lowest terms or before multiplying fractions
Example:
$$\frac{15}{24} = \frac{{}^{1}\cancel{3} \bullet 5}{2 \bullet 2 \bullet 2 \bullet \cancel{3}_{1}} = \frac{5}{8}$$

coefficient ..... a numerical factor in a term of an algebraic expression
Example: In $8a$, the coefficient of a is 8.

common factor ..... a number that is a factor of two or more numbers
Example: 2 is a common factor of 6 and 12.

commutative property ..... the order in which two numbers are added or multiplied does *not* change their sum or product, respectively
Example: $2 + 3 = 3 + 2$ *or* $4 \times 7 = 7 \times 4$

composite number ..... a whole number that has more than two factors
Example: 16 has five factors—1, 2, 4, 8, and 16.

denominator ..... the bottom number of a fraction, indicating the number of equal parts a whole was divided into
Example: In the fraction $\frac{2}{3}$ the denominator is 3, meaning the whole was divided into 3 equal parts.

digit ..... any one of the 10 symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9

distributive property ..... the product of a number and the sum or difference of two numbers is equal to the sum or difference of the two products
Example: $x(a + b) = ax + bx$

exponent (exponential form) ..... the number of times the base occurs as a factor
Example: 2^3 is the exponential form of $2 \times 2 \times 2$. The numeral two (2) is called the *base*, and the numeral three (3) is called the *exponent*.

expression ..... a collection of numbers, symbols, and/or operation signs that stands for a number
Example: $4r^2$; $3x + 2y$; $\sqrt{25}$
Expressions do *not* contain equality (=) or inequality (<, >, ≤, ≥, or ≠) symbols.

factor ..... a number or expression that divides evenly into another number; one of the numbers multiplied to get a product
Example: 1, 2, 4, 5, 10, and 20 are factors of 20 and $(x + 1)$ is one of the factors of $(x^2 - 1)$.

factored form ..... a monomial expressed as the product of prime numbers and variables, where no variable has an exponent greater than 1

FOIL method ..... a pattern used to multiply two binomials; multiply the first, outside, inside, and last terms:

F	First terms
O	Outside terms
I	Inside terms
L	Last terms.

Example:

2 Outside
1 First

$$(a + b)(x - y) = \overset{F}{ax} - \overset{O}{ay} + \overset{I}{bx} - \overset{L}{by}$$

3 Inside
4 Last

fraction ..... any part of a whole
Example: One-half written in fractional form is $\frac{1}{2}$.

greatest common factor (GCF) .... the largest of the common factors of two or more numbers
Example: For 6 and 8, 2 is the greatest common factor.

grouping symbols ..... parentheses ( ), braces { }, brackets [ ], and fraction bars indicating grouping of terms in an expression

integers ..... the numbers in the set
{… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

like terms ..... polynomials with exactly the same variable combinations; terms that have the same variables and the same corresponding exponents
Example: In $5x^2 + 3x^2 + 6$, $5x^2$ and $3x^2$ are like terms with the same variable combinations.

monomial ..... a number, variable, or the product of a number and one or more variables; a polynomial with only *one* term
Examples: $8 \quad x \quad 4c \quad 2y^2 \quad -3 \quad \frac{xyz^2}{9}$

natural numbers (counting numbers) ..... the numbers in the set {1, 2, 3, 4, 5, …}

numerator ..... the top number of a fraction, indicating the number of equal parts being considered
Example: In the fraction $\frac{2}{3}$, the numerator is 2.

opposites ..... two numbers whose sum is zero
Example: $\underset{\text{opposites}}{-5 + 5} = 0$ *or* $\underset{\text{opposites}}{\frac{2}{3} + \left(-\frac{2}{3}\right)} = 0$

order of operations ..... the order of performing computations in parentheses first, then exponents or powers, followed by multiplication and/or division (as read from left to right), then addition and/or subtraction (as read from left to right); also called *algebraic order of operations*

Example: $5 + (12 - 2) \div 2 - 3 \times 2 =$
$5 + 10 \div 2 - 3 \times 2 =$
$5 + 5 - 6 =$
$10 - 6 =$
4

polynomial ..... a monomial or sum of monomials; any rational expression with no variable in the denominator

Examples: $x^3 + 4x^2 - x + 8$ $\quad 5mp^2$
$-7x^2y^2 + 2x^2 + 3$

power (of a number) ..... an exponent; the number that tells how many times a number is used as a factor

Example: In 2^3, 3 is the power.

prime factorization ..... writing a number as the product of prime numbers

Example: $24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3$

prime number ..... any whole number with only two whole number factors, 1 and itself

Example: 2, 3, 5, 7, 11, etc.

product ..... the result of multiplying numbers together

Example: In $6 \times 8 = 48$,
48 is the product.

quotient ..... the result of dividing two numbers

Example: In $42 \div 7 = 6$,
6 is the quotient.

rational expression ..... a fraction whose numerator and/or denominator are polynomials
Examples: $\frac{x}{8}$ $\frac{5}{x+2}$ $\frac{4x^2+1}{x^2+1}$

simplest form
(of an expression) ..... an expression that contains no grouping symbols (except for a fraction bar) and all like terms have been combined
Examples: $6 + y + 3z + 4z = 6 + y + 7z$
$\frac{6xy^2}{5} + \frac{7xy^2}{5} = \frac{13xy^2}{5}$

standard form
(of a quadratic equation) ..... $ax^2 + bx + c = 0$, where a, b, and c are integers (not multiples of each other) and $a > 0$

sum ..... the result of adding numbers together
Example: In $6 + 8 = 14$,
14 is the sum

term ..... a number, variable, product, or quotient in an expression
Examples: z n^2 $8b$ $3r^2h$

trinomial ..... the sum of three monomials; a polynomial with exactly *three* terms
Examples: $x + y + 2$ $m^2 + 6m + 3$
$b^2 - 2bc - c^2$ $8j^2 - 2n + rp^3$

variable ..... any symbol, usually a letter, which could represent a number

whole number ..... the numbers in the set {0, 1, 2, 3, 4, …}

zero property of multiplication or
zero product property ..... for all numbers a and b, if $ab = 0$, then $a = 0$ and/or $b = 0$.

Unit 2: Working with Polynomials

Introduction

We will see that numbers and expressions can be written in a variety of different ways by simplifying and performing operations on polynomials. Reformatting a number does *not* change the value of the number. Simplified expressions often lead us to see important information that unsimplified versions do not.

Lesson One Purpose

- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)

Polynomials

Any **expression** in which the operations are addition, subtraction, multiplication, and division, and all **powers** of the **variables** are **natural numbers** (also known as **counting numbers**). These types of expressions are called **rational expressions**. *Rational expressions* are **fractions** whose **numerator** and/or **denominator** are **polynomials**. Examples of rational expressions are as follows:

$$\frac{x+y}{3x} \qquad x - \frac{1}{x} \qquad \frac{2x+3y}{x-y}$$

Any rational expression with no *variable* in the *denominator* is called a *polynomial*. Examples of polynomials are as follows:

$$x^2 \qquad 7 \qquad 3y^2 - 2y + 1 \qquad x^2y + 2x - y$$

A **term** is a number, variable, **product**, or **quotient** in an expression.

- If a polynomial has only one *term*, we call it a **monomial**, because "mono" means *one*.

  Examples of *monomials*:
  $3 \qquad a^3b \qquad 3xy$

- If a polynomial has exactly two terms, we call it a **binomial**, because "bi" means *two*.

  Examples of *binomials*:
  $x + y \qquad 2x + 3y \qquad 3a^2 - 4b$

- If a polynomial has three terms, we call it a **trinomial**, because "tri" means *three*.

  Examples of *trinomials*:
  $4x + 2y - 3z \qquad x^2 + 3x + 2 \qquad 5ab + 2a - 3b$

Notice that a plus or minus sign separates the terms in all polynomials above. Be careful to notice where those signs occur in the expression.

Note: A polynomial is named *after* it is in its **simplest form**. For example, $3(x + 2y^3)$ must first be *simplified*. Therefore, $3(x + 2y^3) = 3x + 6y^3$, which is a binomial.

Practice

Use the list below to **identify each polynomial**. *Write the word on the line provided.*

binomial	**monomial**	**trinomial**

_____ 1. $3b^2 - b$

_____ 2. $4x^5$

_____ 3. $5t^2 - 3t^5$

_____ 4. $5x^3 - 4x^2 + 3x$

_____ 5. $3r^2st^2$

_____ 6. $x - y + 3$

Practice

Use the list below to **identify each polynomial**. *Write the word on the line provided.*

binomial	**monomial**	**trinomial**

_____ 1. $3x^3 - 2x^2 + 1$

_____ 2. $4xy^2z$

_____ 3. $a - b + 2$

_____ 4. $2a^2 - a$

_____ 5. $6b^2$

_____ 6. $3x^2 - 5y^2$

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____	1. a monomial or sum of monomials	A. binomial
_____	2. a polynomial with only *one* term	B. expression
_____	3. an exponent; the number that tells how many times a number is used as a factor	C. monomial
_____	4. any symbol, usually a letter, which could represent a number	D. natural numbers
_____	5. a polynomial with exactly *three* terms	E. polynomial
_____	6. a collection of numbers, symbols, and/or operation signs that stands for a number	F. power (of a number)
_____	7. a polynomial with exactly *two* terms	G. trinomial
_____	8. the numbers in the set {1, 2, 3, 4, 5, …}	H. variable

Lesson Two Purpose

- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil and calculator. (MA.A.3.4.3)
- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)

Addition and Subtraction of Polynomials

Polynomials with *exactly* the same variable combinations can be added or subtracted. For example, $7xy$ and $3xy$ have the same variable combination. We call these **like terms**.

$$7xy + 3xy = 10xy \qquad \text{and} \qquad 7xy - 3xy = 4xy$$

A polynomial is in *simplest form* if it contains no **grouping symbols** (except a *fraction* bar) and all *like terms* have been combined.

Polynomials can be arranged in any order. In **standard form**, polynomials are arranged from *left to right*, from *greatest to least* degree of *power*. For example:

$$x^7 - x^2 + 8x$$

Polynomials can be added or subtracted in vertical (↕) or horizontal (↔) form.

Addition

vertical form

$(3y^2 + 2y + 3) + (y^2 + 1) =$

Align like terms in columns and add.

$$\begin{array}{rl} & 3y^2 + 2y + 3 \\ (+) & \underline{y^2 \qquad\quad + 1} \\ & 4y^2 + 2y + 4 \end{array}$$

write degrees of powers left to right from greatest to least

align like terms

add like terms

horizontal form

$(3y^2 + 2y + 3) + (y^2 + 1) =$

Regroup and add like terms.

$(3y^2 + y^2) + (2y) + (3 + 1) =$ ← group like terms

$4y^2 + 2y + 4$ ← add like terms

Subtraction

You subtract a polynomial by adding its **additive inverse** or **opposite**. To do this, multiply each term in the *subtracted* polynomial by -1 and add.

polynomial	additive inverse
$-8y + 4x$	$8y - 4x$
$3q^2 - 6r + 11$	$-3q^2 + 6r - 11$
$2a + 7b - 3$	$-2a - 7b + 3$

vertical form

$(3y^2 - 2y + 3) - (y^2 - 1)$

Align like terms in columns and subtract by adding the additive inverse.

write degrees of powers left to right from greatest to least

$$\begin{array}{rrrr} & 3y^2 & -2y & +3 \\ (-) & y^2 & & -1 \\ \hline \end{array}$$

align like terms

add additive inverse

add like terms

$$\begin{array}{rrrr} & 3y^2 & -2y & +3 \\ (+) & -y^2 & & +1 \\ \hline & 2y^2 & -2y & +4 \end{array}$$

horizontal form

$(3y^2 - 2y + 3) - (y^2 - 1)$

Subtract by adding *additive inverse* and group like terms.

$[3y^2 + (-2y) + 3] + [(-y^2) + 1] =$ ← add additive inverse of $2y$, which is $-2y$, and $y^2 - 1$, which is $-y^2 + 1$

$[3y^2 + (-y^2)] + (-2y) + (3 + 1) =$ ← group like terms

$2y^2 + -2y + 4 =$ ← add like terms

vertical form

Subtract $2t^2 - 3t + 4$ from the **sum** of $t^2 + t - 6$ and $3t^2 + 2t - 1$.

$(t^2 + t - 6) + (3t^2 + 2t - 1) - (2t^2 - 3t + 4)$

write degrees of powers left to right from greatest to least

$$\begin{array}{rr} & t^2 + t - 6 \\ & 3t^2 + 2t - 1 \\ (-) & \underline{2t^2 - 3t + 4} \end{array}$$

align like terms

add additive inverse

$$\begin{array}{rr} & t^2 + t - 6 \\ & 3t^2 + 2t - 1 \\ (+) & \underline{-2t^2 + 3t - 4} \\ & 2t^2 + 6t - 11 \end{array}$$

horizontal form

Subtract $2t^2 - 3t + 4$ from the *sum* of $t^2 + t - 6$ and $3t^2 + 2t - 1$.

$(t^2 + t - 6) + (3t^2 + 2t - 1) - (2t^2 - 3t + 4)$

$\boxed{t^2} + t - 6 \boxed{+ 3t^2} + 2t - 1 \boxed{- 2t^2} + 3t - 4 =$

add additive inverse of $-(2t^2 - 3t + 4)$, which is $-2t^2 + 3t - 4$

$2t^2 + 6t - 11 =$

combine like terms

Practice

Write each **expression** *in* **simplest form**. *Use either the* **horizontal** *or* **vertical** *form. Refer to examples on pages 89-91 as needed. Show* **essential steps**.

Remember: Write answers with the degree of powers arranged from left to right and from greatest to least.

Example: $(3y^2 - 2y + 3) - (y^2 - 1) =$

horizontal form | **vertical form**

$(3y^2 - 2y + 3) - (y^2 - 1) =$

add additive inverse and group like terms

$(3y^2 + -y^2) + (-2y) + (3 + 1) =$

$2y^2 + -2y + 4 =$

add like terms

$$\begin{array}{rl} & 3y^2 - 2y + 3 \\ (+) & \underline{-y^2 \qquad\quad + 1} \\ & 2y^2 - 2y + 4 \end{array}$$

1. $3ab^2 - 5a^2b + 5ab^2 =$

2. $(2x^2 - 3x + 7) - (3x^2 + 3x - 5) =$

3. $(2x^3 - 3x^2 + 2x) + (4x - 2x^2 - 3x^3) =$

4. $(4a^2 + 6a - 6) + (3a^2 - 2a + 4) - (5a^2 - 5a - 9) =$

5. $(-3y^3 + 4y^2 + 6y) - (y^3 - 2y^2 + y + 6) + (4y^3 + 2y^2 - 4y - 1) =$

6. $(a^3 - 3a^2b - 4ab^2 + 6b^3) - (a^3 + a^2b - 2ab^2 - 5b^3) =$

7. $3a + [5a - (a + 3)] =$

8. $[x^2 - (2x - 3)] - [2x^2 + (x - 2)] =$

9. $5 - [3y + (y - 2) - 1] =$

10. $y - \{y - [x - (2x - y)] + 2y\} =$

Example: Subtract $2t^2 - 3t + 4$ from the sum of $t^2 + t - 6$ and $3t^2 + 2t - 1$.

horizontal form

$$(t^2 + t - 6) + (3t^2 + 2t - 1) - (2t^2 - 3t + 4) =$$

add additive inverse

$$t^2 + t - 6 + 3t^2 + 2t - 1 - 2t^2 + 3t - 4 =$$

group like terms and add

$$2t^2 + 6t - 11$$

vertical form

$$\begin{array}{rrrr} & t^2 + & t - & 6 \\ & 3t^2 + & 2t - & 1 \\ (+) & -2t^2 + & 3t - & 4 \\ \hline & 2t^2 + & 6t - & 11 \end{array}$$

11. Subtract $4x^2 - 3x + 3$ from the sum of $x^2 - 2x - 3$ and $x^2 - 4$.

12. Subtract $2t^2 - 3t + 5$ from the sum of $4t^2 - 3t + 4$ and $-t^2 + 5t + 7$.

Practice

Write each **expression** *in* **simplest form**. *Use either the* **horizontal** *or* **vertical** *form. Refer to examples on previous practice and pages 89-91 as needed. Show* **essential steps**.

1. $5xy^2 + 2x^2y - 6xy^2 =$

2. $(6a^2 - 4a - 3) - (5a^2 + 2a + 1) =$

3. $(3y^3 - 4y^2 + 9y) + (5y^3 - 6y^2 + 6) =$

4. $(8x^2 + 2x - 6) - (4x^2 - 3x + 9) + (5x^2 + 2x - 3) =$

5. $(8a^3 - 2a^2 + 3a) - (9a^3 + 5a - 4) + (6a^2 - 8a + 5) =$

6. $(x^3 - 4x^2y - 6xy^2 + 2y^3) - (x^3 + 6x^2y - 9xy^2 + 6y^3) =$

7. $5x + [3x - (x + 2)] =$

8. $b^2 - [4b - (b + 6)] =$

9. $7 - [4x + (x - 2) - 4] =$

10. $x - \{2x - [x + (2x - y)] + 5y\} =$

11. Subtract $3x^2 + 2x - 1$ from the sum of $8x^2 - 6x + 9$ and $x^2 - 8$.

12. Subtract $2a^2 - 6a + 4$ from the sum of $a^2 + 4$ and $4a^2 - 9a + 8$.

Lesson Three Purpose

- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil and calculator. (MA.A.3.4.3)
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Multiplying Monomials

First Law of Exponents

a^5

exponent (exponential form) is the number of times the **base (of an exponent)** occurs as a **factor**

base (of an exponent) is the number that is used as a *factor* in *exponential form*

For example, a^5a^3 means a^5 times a^3 or $(aaaaa)(aaa)$. By *counting* the number of a's, which is 8, you can see that

$$a^5a^3 = a^8.$$

This is an example of the *first law of exponents*, which states that $a^xa^y = a^{x+y}$.

Below are a couple of other examples.

$$x^2x^3 = x^5 \qquad xx^4x^5 = x^{10} \qquad b^4b^2b^3 = b^9$$

When there are **coefficients** (the **digit** you multiply the variable by) other than 1, you must multiply those first and then use the first law of exponents. In the expression $2x^2y$, the *digit* 2 is the *coefficient*. In the expression $3xy^4$, the digit 3 is the coefficient.

First Law of Exponents—Product of Powers

You *multiply exponential forms* with the same base by *adding the exponents*.

$$4^3 \bullet 4^2 = 4^{3+2} \text{ or } 4^5$$

$$x^a \bullet x^b = x^{a+b}$$

If we multiply $2x^2y$ and $3xy^4$, this would be

$$(2x^2y)(3xy^4) =$$

$$\underbrace{2 \bullet 3}_{6} \bullet \underbrace{x^2 \bullet x}_{x^3} \bullet \underbrace{y \bullet y^4}_{y^5} =$$

$$6x^3y^5$$

- The coefficients are multiplied.

  $2 \bullet 3 = 6$

- The exponents are added.

  $x^2 + x = x^3$ and $y + y^4 = y^5$

If we multiply $7b$ and $-b^3$, this would be

$$(7b)(-b^3) =$$
$$-7b^4$$

- The coefficients are multiplied.

  $$7 \bullet -1 = -7$$

  (The digit -1 is understood to be in front of the variable $-b^3$.)

- The exponents are added.

  $$b + -b^3 = -b^4$$

Remember: Use the rules for the **order of operations**. Complete multiplication as it occurs, from left to right, including all understood coefficients.

Example

$-x^3(x^4)(5x)(-2x^4) =$

add the exponents

$$\overbrace{-x^3(x^4)}^{3+4=7}\overbrace{(5x)}^{7+1=8}\overbrace{(-2x^4)}^{8+4=12=x^{12}} = 10x^{12}$$

$-1 \bullet 1 = -1 \quad -1 \bullet 5 = -5 \quad -5 \bullet -2 = 10$

multiply the coefficients left to right

Practice

Write each **product** *as a* **polynomial** *in* **simplest form**.

Remember: Multiply the coefficients and add the exponents.

Example: $(7a^2)(5a^3b^4) =$
$35a^5b^4$

1. $(6t)(-3t^3) =$

2. $(5x)(-x^4) =$

3. $(-6r^2s)(4r^2s^3) =$

4. $(-5a)(ab^3)(-3a^2bc) =$

5. $(y^2z)(-3x^2z^2)(-y^4z) =$

6. $-a^2(ab^2)(3a)(-2b^3) =$

7. $(-t)^2(2t^2)(5t)^2 =$

Hint: Notice with $(-t)^2$ and $(5t)^2$, the exponent 2 is placed on the *outside* of the grouping symbols, the parentheses. Use the **distributive property** and raise every term in the parentheses to the exponent.

Example: $(-t)^2 = t^2$
$(5t)^2 = 25t^2$

8. $(3x^2)(-5x^3y^2)(0)(-4y)^2 =$

Hint: Notice the zero (0). The **zero property of multiplication**, also known as the **zero product property**, states that any number multiplied by 0 is 0.

Zero Property of Multiplication *or* Zero Product Property
For all numbers a and b, if $ab = 0$, then $a = 0$ and/or $b = 0$.

Practice

Write each **product** *as a* **polynomial** *in* **simplest form**.

1. $(8x)(-2x^2) =$

2. $(5a)(-a^6) =$

3. $(-4x^2y)(3x^3y^2) =$

4. $(-6b)(ab^4)(-4a^2bc^2) =$

5. $(x^3y^2)(-2x^2y)(-x^4y^2) =$

6. $-s^3(s^2t^2)(4s)(-2t^4) =$

7. $(-a)^2(4a^2)(3a)^2 =$

8. $(6x)^2(-2x^2y^3)(0)(-2x)^2 =$

Lesson Four Purpose

- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil and calculator. (MA.A.3.4.3)
- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)

Dividing Monomials

Second Law of Exponents

When dividing monomials it is important to remember that

$$\frac{a^5}{a^3} = \frac{aaaaa}{aaa}.$$

It is also important to remember that

$\frac{a}{a} = 1$ because

$a^0 = 1.$

Therefore, the three a's in the *denominator* **cancel** three of the five a's in the *numerator*. This leaves $a \bullet a$ or a^2 in the numerator and 1 in the denominator.

Remember: To *cancel* means to divide a numerator (the top part of the fraction) and a denominator (the bottom part of the fraction) by a **common factor**. This is done in order to write the fraction in lowest terms or before multiplying the fractions.

numerator

$$\frac{\cancel{a}\cancel{a}\cancel{a}aa}{\cancel{a}\cancel{a}\cancel{a}} = \frac{aa}{1} \text{ or } \frac{a^2}{1} = a^2$$

denominator

Another way to look at this is

$$\frac{a^5}{a^3} = a^{5-3} = a^2.$$

This is an example of the *second law of exponents*, which states that

$$\frac{a^x}{a^y} = a^{x-y}$$

as long as $a \neq 0$.

Second Law of Exponents

You *divide exponential forms* by *subtracting the exponents*.

$$9^7 \div 9^3 = 9^{7-3} = 9^4$$

Remember: The *fraction bar* represents division. So, $\frac{8^4}{8^2}$ means $8^4 \div 8^2$.

$$\frac{9^7}{9^3} = 9^{7-3} = 9^4$$

$$\frac{a^m}{a^n} = a^{m-n}$$

If the exponents are the same,

$$\frac{a^x}{a^x} = 1 \text{ because}$$

$$\frac{a^x}{a^x} = a^{x-x} = a^0 = 1.$$

Any number (except zero) raised to the zero power is equal to 1.

$$a^0 = 1$$

Example

$$\frac{x^4b^3}{xb^3} =$$

$$x^{4-1} \bullet \frac{b^3}{b^3} = \longleftarrow \frac{\cancel{b^3}}{\cancel{b^3}} = 1$$

$$x^3 \bullet 1 =$$

$$x^3$$

When there are coefficients with variables, simply reduce those as you do when working with fractions.

Example

$$\frac{12a^3b^5}{-4ab^3} = \longleftarrow \frac{\cancel{12}^{\,3}}{\cancel{4}_{\,-1}} = \frac{3}{-1} = -3$$

$$-3a^{3-1}b^{5-3} =$$

$$-3a^2b^2$$

Practice

Write each **quotient** *as a* **polynomial** *in* **simplest form**. *Refer to examples on pages 105-107 as needed. Show* **essential steps**.

1. $\frac{6x^3y^4}{3xy} =$

2. $\frac{14c^4d^3}{-7c^4d^2} =$

3. $\frac{100m^5n}{-20m^3n} =$

4. $\frac{-22a^2bc^5}{11abc} =$

5. $\dfrac{12r^2st^3}{-3rst^3} =$

6. $\dfrac{a^5b^6c^7}{a^4b^3c^2} =$

7. $\dfrac{(t+4)^5}{(t+4)^2} =$

Hint: Notice that the exponents 5 and 2 are on the *outside* of the grouping symbols, the parentheses. Just subtract the exponents. Do *not* raise each term in the parentheses to the exponent.

8. $\dfrac{9(x-3)^3}{-3(x-3)^2} =$

Practice

Write each **quotient** *as a* **polynomial** *in* **simplest form**. *Refer to examples on pages 105-107 as needed. Show* **essential steps**.

1. $\dfrac{8a^2b^4}{4ab^2} =$

2. $\dfrac{16x^5y^4}{-8x^3y} =$

3. $\dfrac{-36a^2b^5c^4}{3ab^2c} =$

4. $\dfrac{48x^2yz^3}{12xyz} =$

5. $\dfrac{20a^2bc^3}{10abc^2} =$

6. $\dfrac{x^5y^7z^6}{x^2y^2z^6} =$

7. $\dfrac{(x+1)^5}{(x+1)^3} =$

8. $\dfrac{10(x+7)^4}{-5(x+7)^2} =$

Lesson Five Purpose

- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil and calculator. (MA.A.3.4.3)
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Multiplying Polynomials

Using the Distributive Property to Multiply a Monomial and a Trinomial

Multiplying a *monomial* and a *polynomial* is simply an extension of the *distributive property*. Make sure that *every term in the parentheses* is multiplied by the *term in front of the parentheses*.

$$3x(2a + 3b - 4c) =$$
$$6xa + 9xb - 12xc$$

multiply every term in the parentheses by the term $3x$ in front of the parentheses

Typically, mathematicians like to put things in order and would rearrange the variables in the answer above so that the variables in each term are *alphabetical*. Therefore, the final answer would be as follows.

$$6xa + 9xb - 12xc =$$
$$6ax + 9bx - 12cx$$

Using the FOIL Method to Multiply Two Binomials

When we multiply two polynomials, we extend the distributive property even further to make sure that every term in the *first* set of parentheses is multiplied by every term in the *next* set of parentheses.

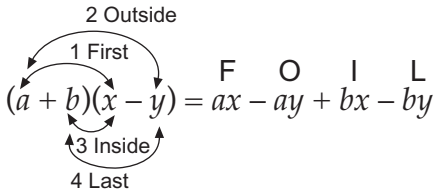
Look carefully at the product below.

$$(a + b)(x - y) = ax - ay + bx - by$$

Notice that both x and y were multiplied by a, and then by b. This is called the **FOIL method** because

- the two **F**irst terms (a and x) are multiplied,
- then the two **O**utside terms (a and -y) are multiplied,
- then the two **I**nside terms (b and x) are multiplied and lastly,
- the two **L**ast terms (b and -y) are multiplied together.

F	**F**irst terms
O	**O**utside terms
I	**I**nside terms
L	**L**ast terms



It is important to be *orderly* when you multiply to insure that you don't leave out a step. Also, be very careful to watch the positive (+) and negative (–) signs as you work.

There are some special patterns that often occur. Knowing these may help you.

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

F + O + I + L

$$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$$

F + O + I + L

$$(a-b)(a+b) = a^2 - b^2$$

F + O + I + L

Alert!

$(a+b)^2 \neq a^2 + b^2$ ← To write this expression in simplest form, the power of 2 is not simply distributed over $a + b$. Instead…

$(a+b)^2 = (a+b)(a+b)$ ← $(a + b)^2$ is multiplied by itself, $(a + b)(a + b)$.

Using the Distributive Property to Multiply Any Two Polynomials

Let's look at using the distributive property to do the following.

- multiply a binomial and a trinomial in horizontal form
- multiply two trinomials in horizontal form
- multiply polynomials in vertical form

Example 1

Finding the product of a binomial and a trinomial in **horizontal form**

$$(2a + 5)(3a^2 - 8a + 7) =$$

$$(2a + 5)(3a^2 - 8a + 7) =$$

$$2a(3a^2 - 8a + 7) + 5(3a^2 - 8a + 7) =$$

distributive property

$$(6a^3 - 16a^2 + 14a) + (15a^2 - 40a + 35) =$$

combine

$$6a^3 \boxed{- 16a^2} + 14a \boxed{+ 15a^2} - 40a + 35 =$$

combine

combine like terms

$$6a^3 - a^2 - 26a + 35$$

Example 2

Finding the product of two trinomials in **horizontal form**

$$(b^2 + 4b - 5)(3b^2 - 7b + 2) =$$

$$(b^2 + 4b - 5)(3b^2 - 7b + 2) =$$

$$b^2(3b^2 - 7b + 2) + 4b(3b^2 - 7b + 2) - 5(3b^2 - 7b + 2) =$$

$$(3b^4 - 7b^3 + 2b^2) + (12b^3 - 28b^2 + 8b) - (15b^2 - 35b + 10) =$$

$$3b^4 - 7b^3 + 2b^2 + 12b^3 - 28b^2 + 8b - 15b^2 + 35b - 10 =$$

$$3b^4 + 5b^3 - 41b^2 + 43b - 10$$

distributive property

combine like terms

Example 3

Finding the product of polynomials in **vertical form**

$$(c^3 - 8c^2 + 9)(3c + 4) =$$

Note: There is no c term in $c^3 - 8c^2 + 9$, so $0c$ is used as a placeholder.

$$\begin{array}{rrrrr}
c^3 - & 8c^2 + & 0c + & 9 & \\
(x) & & 3c + & 4 & \\
\hline
4c^3 - & 32c^2 + & 0c + & 36 & \\
3c^4 - & 24c^3 - & 0c^2 + & 27c & \\
\hline
3c^4 - & 20c^3 - & 32c^2 + & 27c + & 36
\end{array}$$

Practice

Write each **expression** *as a* **polynomial** *in* **simplest form**. *Refer to examples on pages 112-116 as needed. Show* **essential steps**.

Example: $-3(2x + 4y - z) =$ distributive property

$-6x - 12y + 3z$

1. $2a(a + 3b) =$

2. $-5x(3x - 2y + 6z) =$

Example: $(x - 4)^2 =$

$(x - 4)(x - 4) =$ FOIL method

$x^2 - 4x - 4x + 16 =$

$x^2 - 8x + 16$

3. $(x + 4)^2 =$

4. $(x + 8)^2 =$

Example: $(y - 3)(y + 4) =$ ← FOIL method

$y^2 + 4y - 3x - 12 =$

$y^2 + y - 12$

5. $(a - 3)(a + 6) =$

6. $(x + 2)(x - 2) =$

7. $(2x + 5)(3x - 6) =$

8. $(3t - 1)(3t + 5) =$

9. $(3g - 4)(2g - 3) =$

Example:

horizontal form

$$(x - 4)(x^2 + 2x - 3) =$$

distributive property

$$x^3 + 2x^2 - 3x - 4x^2 - 8x + 12 =$$
$$x^3 - 2x^2 - 11x + 12$$

vertical form

$$\begin{array}{rr} & x^2 + 2x - 3 \\ (x) & \underline{\qquad\qquad x - 4} \\ & -4x^2 - 8x + 12 \\ x^3 + 2x^2 - 3x & \\ \hline x^3 - 2x^2 - 11x + 12 & \end{array}$$

Notice the order of the terms in the answer above. The values of the exponents are in *decreasing* order: x^3, x^2, x^1, x^0.

10. $(x + 2)(x^2 - 2x + 3) =$

11. $(x^2 - 3x + 5)(x - 6) =$

12. $(2a^2 - 3a + 1)(3a^2 + 2a + 1) =$

Practice

Write each **expression** *as a* **polynomial** *in* **simplest form**. *Refer to examples on pages 112-116 as needed. Show* **essential steps**.

Use the **distributive property**.

1. $6s(s^2 - 3s + 2) =$

2. $2y^3(3y^2 - 4y + 7) =$

Use the **FOIL method**.

3. $(x - 3)^2 =$

4. $(x - 10)^2 =$

5. $(b + 5)(b + 4) =$

6. $(c - 5)(c + 5) =$

7. $(2z - 3)(4z + 2) =$

Use the **distributive property**.

8. $(b + 5)(b^2 + 4b - 9) =$

9. $(y^2 - 3y + 7)(y^2 + 4) =$

10. $(a + 3)(a - 4)(a - 5) =$

    Hint: Multiply the first two binomials. Then multiply that product by the third binomial. Use either the vertical or horizontal form to do this.

Lesson Six Purpose

- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil and calculator. (MA.A.3.4.3)
- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)

Factoring Polynomials

If we look at the product abc, we know a, b, and c are *factors* of this product. In the same way, 2 and 3 are *factors* of 6. Other factors of 6 are 6 and 1.

factors of $abc = a$, b, and c

factors of 6 = 1, 2, 3, and 6

Some numbers, like 5, have no factors other than the number itself and the number 1. These numbers are called **prime numbers**. A *prime number* is any **whole number** {0, 1, 2, 3, 4, …} with only two factors, 1 and itself. The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19.

prime numbers < 20 = 2, 3, 5, 7, 11, 13, 17, and 19

Natural numbers greater than 1, which are *not* prime, are called **composite numbers**. A *composite number* is a whole number with more than two factors. For example, 16 has five factors, 1, 2, 4, 8, and 16. Therefore, 4, 6, 8, 9, 10, 12, 14, 15, 16, and 18 are the composite numbers less than 20.

composite numbers < 20 = 4, 6, 8, 9, 10, 12, 14, 15, 16, and 18

Every composite number can be written as a *product* of prime numbers. We can find this **prime factorization** by factoring the factors and repeating this process until all factors are primes.

For example, find the *prime factorization* of 24 and express it in completely **factored form**.

Factoring a Positive Number—
numbers greater than zero

Method One

Use a factor tree.

```
      24
     /  \
   6  •  4
  / \    / \
3 • 2 • 2 • 2
```

24 = 3 • 2 • 2 • 2

or

Method Two

24
6 • 4
(3 • 2)(2 • 2)
3 • 2 • 2 • 2

Factoring a Negative Number—
numbers less than zero

Method One

```
      -24
     /  \
   -6  •  4
  / \    / \
-3 • 2 • 2 • 2
```

-24 = -3 • 2 • 2 • 2

or

Method Two

-24
-6 • 4 =
(-3 • 2)(2 • 2) =
-3 • 2 • 2 • 2

Alternate Method to Factoring a Positive Number

Here is an alternate method to factoring a positive number called *upside-down dividing*. Divide by prime numbers starting with the number 2.

```
2)24
2)12
2) 6
3) 3
   1
```

To factor polynomials, you must look carefully at each term and decide if there is a factor that is *common* to each term. If there is, we basically "undistribute" or *factor out* that **greatest common factor (GCF)**. Look at the example below.

$6x^3 - 12x^2 + 3x =$ ← Notice that each term can be divided by 3 and x. So, $3x$ is the *greatest factor* these terms have in *common*. Therefore, $3x$ is the *GCF* of $6x^3 - 12x^2 + 3x$.

$3x(2x^2 - 4x + 1)$ ← undistribute the $3x$

All of the terms and symbols must be written to make sure that your new expression is *exactly* equal to the original one. You can check your work by distributing the $3x$ to everything within the parentheses to see if it matches the original expression.

$$3x(2x^2 - 4x + 1) = 6x^3 - 12x^2 + 3x$$

Remember: $(a + b) = (b + a)$ — The **commutative property** of addition—numbers can be added in any order and the sum will be the same.

Alert!

$(a - b) \neq (b - a)$ — The same is *not* true for $a - b$. The *commutative property* does *not* work with subtraction.

$a - b$ does *not* equal $b - a$

$(a - b) = -1(b - a)$ — $a - b$ is understood as $a - {}^{+}b$, therefore,

$a - b$ equals $-1(b - a)$

Practice

Express each **integer {… , -3, -2, -1, 0, 1, 2, 3, …}** *in completely* **factored form**. *If the integer is a* **prime number**, *write* **prime** *after the equal sign.*

1. 8 =

2. 18 =

3. -16 =

4. 23 =

5. 56 =

Factor *the following completely. Show* **essential steps**.

Example: $18x^3y - 24x^2y^2 =$

$6x^2y(3x - 4y)$

Find the **GCF**, which is $6x^2y$, and *undistribute* it.

6. $3a - 9 =$

7. $2x^2y^2 + 3xy - 4xy^3 =$

8. $3m^4 + 6m^3 - 12m^2 =$

9. $ay^3b + a^2y^2 + ab =$

Example: $x\boxed{(b+2)} - 7\boxed{(b+2)} =$

$(b+2)(x-7)$

10. $a(a+3) - 6(a+3) =$

11. $2x(x+5) - 3(x+5) =$

12. $5(y-7) + z(y-7) =$

Practice

Express each **integer {… , -3, -2, -1, 0, 1, 2, 3, …}** *in completely* **factored form**.
If the integer is a **prime number**, *write* **prime** *after the equal sign.*

1. 12 =

2. 15 =

3. -25 =

4. 31 =

5. 72 =

Factor *the following completely. Show* **essential steps**.

6. $4b^2 + 12b =$

7. $y^4 - y^3 + y =$

8. $15r^2s + 9rs^2 - 12rs =$

9. $16x^2yz^3 + 8x^3y^2z^2 - 24x^4y^2z =$

10. $y(y - 4) + 4(y - 4) =$

11. $5x(3a + 1) - 4(3a + 1) =$

12. $3a(2x - y) + 4a(2x - y) =$

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. the largest of the common factors of two or more numbers

_____ 2. any whole number with only two whole number factors, 1 and itself

_____ 3. the order in which two numbers are added or multiplied does *not* change their sum or product, respectively

_____ 4. a monomial expressed as the product of prime numbers and variables, where no variable has an exponent greater than 1

_____ 5. writing a number as the product of prime numbers

_____ 6. a whole number that has more than two factors

A. commutative property

B. composite number

C. factored form

D. greatest common factor (GCF)

E. prime factorization

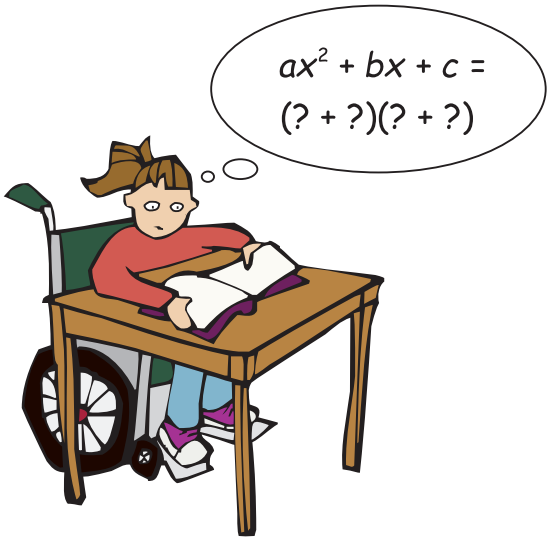
F. prime number

Lesson Seven Purpose

- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, absolute value, *and logarithms*. (MA.A.1.4.4)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil and calculator. (MA.A.3.4.3)
- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)

Factoring Quadratic Polynomials

Polynomials that are written in the format $ax^2 + bx + c$ can be factored into two binomials. The following six-step method may help, especially if you have had difficulty with factoring in the past.



Example 1

Format $ax^2 + bx + c$

Step 1 $6x^2 + 17x + 5$ ← Write the problem. Factor out common factors, if there are any. Identify *a*, *b*, and *c*.
a = 6, *b* = 17, and *c* = 5

Step 2 $ac = 6 \bullet 5$
$= 30$ ← Multiply *a* and *c*.

Step 3 $6x^2 + 2x + 15x + 5$ ← Rewrite the problem using factors *ac*. The factors you choose must combine (add or subtract) to *equal the middle term*.

Alert! $2x + 15x = 17x$, which is the *same* as the *original* middle term.

Step 4 $(6x^2 + 2x) + (15x + 5)$ ← Group the first two terms and the last two terms.

Step 5 $2x(3x + 1) + 5x(3x + 1)$ ← Factor out the greatest common factor for each term. You will always be left with a matching pair of factors. Notice the $(3x + 1)$s. If you do **not** have a matching pair, double-check your work at this point!

Step 6 $(3x + 1)(2x + 5)$ ← Write down the common factor $(3x + 1)$. Then write the "leftovers" in parentheses. You have succeeded!

The next example shows how to handle minus signs. Watch carefully!

Example 2

Format	$ax^2 + bx + c$	
Step 1	$4x^2 - 5x + 1$	← Write the problem. Factor out common factors, if there are any. Identify *a*, *b*, and *c*. *a* = 4, *b* = -5, and *c* = 1
Step 2	$ac = 4 \bullet 1$ $= 4$	← Multiply *a* and *c*.
Step 3	$4x^2 - 4x - x + 1 =$ $4x^2 + -4x + -x + 1 =$	← Rewrite the problem using factors *ac*. The factors you choose must combine (add or subtract) to *equal the middle term*.
Step 4	$(4x^2 + -4x) + (-x + 1) =$	← Group the first two terms and the last two terms. *If the second term in step 3 is followed by a minus sign, this* **requires** *a sign change to each term in that set.*
Step 5	$4x(x - 1) + -1(x - 1) =$	← Factor out the greatest common factor for each term. You must always have a common factor, even if it is only a 1. You will always be left with a matching pair of factors. Notice the $(x - 1)$s. If you do **not** have a matching pair, double-check your work at this point!
Step 6	$(x - 1)(4x - 1)$	← Write down the common factor $(x - 1)$. Then write the "leftovers" in parentheses. You have succeeded!

Now, you try one!

Example 3

Format $ax^2 + bx + c$

Step 1 $4x^2 + 4x - 3$ ← Write the problem. Factor out common factors, if there are any. Identify *a*, *b*, and *c*.
a = ____ , *b* = ____ , and *c* = ____

Step 2 ac = _____
= _____ ← Multiply *a* and *c*.

Step 3 _____ ← Rewrite the problem using factors *ac*. The factors you choose must combine (add or subtract) to *equal the middle term*.

Step 4 _____ ← Group the first two terms and the last two terms. *If the second term in step 3 is followed by a minus sign, this* **requires** *a sign change to each term in that set.*

Step 5 _____ ← Factor out the greatest common factor for each term. You must always have a common factor, even if it is only a 1. You will always be left with a matching pair of factors. Notice the $(x - 1)$s. If you do **not** have a matching pair, double-check your work at this point!

Step 6 _____ ← Write down the common factor. Then write the "leftovers" in parentheses.

Use FOIL to check your answer. If your answer is $(2x + 3)(2x - 1)$, you have succeeded!

Now you are ready to practice some problems on your own.

Practice

Factor *completely. Show* **essential steps**.

Format $ax^2 + bx + c$

Example: $8x^2 + 12x - 8 =$ ← $a = 2$, $b = 3$, and $c = -2$

$4(2x^2 + 3x - 2) =$ ← $ac = -4$

$4(2x^2 + 4x - x - 2) =$

$4[2(x + 2) - 1(x + 2)] =$

$4[(x + 2)(2x - 1)]$

1. $6b^2 + 17b + 5 =$

2. $3x^2 - 8x + 5 =$

3. $3a^2 + 7a - 6 =$

4. $2y^2 + 7y + 5 =$

5. $8x^2 - 6x - 9 =$

6. $10a^2 + 11a - 6 =$

7. $3x^2 + 4x + 1 =$

8. $4a^2 - 5a + 1 =$

9. $2r^2 + 3r - 2 =$

Practice

Factor *completely. Show* **essential steps**.

Format $ax^2 + bx + c$

Example: $x^2 - 2x - 3 =$ ← $a = 1$, $b = -2$, and $c = -3$

$x^2 - 3x + x - 3 =$ ← $ac = -3$

$(x^2 - 3x) + (x - 3) =$

$x(x - 3) + 1(x - 3) =$

$(x - 3)(x + 1)$

1. $a^2 - a - 6 =$

2. $y^2 + 7y + 12 =$

3. $x^2 + 7x + 10 =$

4. $a^2 - 2a + 15 =$

5. $x^2 + 6x + 5 =$

Take opportunities to practice factoring problems like the ones in this practice and use the factors of the middle term with trial and error tactics.

Practice

Factor *completely. Show* **essential steps**.

Format $ax^2 + bx + c$

Example: $x^2 - 4 =$ ← insert a middle term of $0x$

$x^2 + 0x - 4 =$ ← $a = 1$, $b = 0$, and $c = -4$; $ac = -4$

$x^2 + 2x - 2x - 4 =$ ← rewrite b as $+2x - 2x$

$(x^2 + 2x) - (2x + 4) =$ ← group the first two and last two terms

Remember: If the second term is followed by a minus sign, this requires a sign change to each term in that set.

$x(x + 2) - 2(x + 2) =$ ← take out common factors

$(x + 2)(x - 2)$ ← rewrite using common factors

1. $a^2 - 16 =$

2. $x^2 - 9 =$

3. $b^2 - 25 =$

4. $y^2 - 81 =$

5. $x^2 - 36y^2 =$

Notice that the final terms in the problems above were all *perfect squares* and the answers fit the pattern $a^2 - b^2 = (a + b)(a - b)$. Use this shortcut whenever possible. However, if you are unsure, you can always use the six-step method used in the previous practices.

Remember: A *perfect square* is a number whose *square root* is a whole number.
Example: 25 is a perfect square because 5 x 5 = 25.

Practice

Factor *completely. Show* **essential steps**.

Format $ax^2 + bx + c$

Example: $8x^2 + 12x - 8 =$ ← $a = 2$, $b = 3$, and $c = -2$
$4(2x^2 + 3x - 2) =$ ← $ac = -4$
$4(2x^2 + 4x - x - 2) =$
$4[2(x + 2) - 1(x + 2)] =$
$4[(x + 2)(2x - 1)]$

1. $2x^2 + 3x - 20 =$

2. $15x^2 + 13x + 2 =$

3. $6x^2 - 7x - 10 =$

4. $x^2 + 6x + 8 =$

5. $x^2 + x - 12 =$

Practice

Factor *completely. Show* **essential steps**.

Format $ax^2 + bx + c$

Example: $x^2 - 2x - 3 =$ ← $a = 1$, $b = -2$, and $c = -3$
$x^2 - 3x + x - 3 =$ ← $ac = -3$
$(x^2 - 3x) + (x - 3) =$
$x(x - 3) + 1(x - 3) =$
$(x - 3)(x + 1)$

1. $x^2 - 3x - 4 =$

2. $x^2 - 3x + 2 =$

3. $x^2 - 8x + 15 =$

Example: $x^2 - 4 =$ ← insert a middle term of $0x$

$x^2 + 0x - 4 =$ ← $a = 1$, $b = 0$, and $c = -4$; $ac = -4$

$x^2 + 2x - 2x - 4 =$ ← rewrite b as $+2x - 2x$

$(x^2 + 2x) - (2x + 4) =$ ← group the first two and last two terms

Remember: If the second term is followed by a minus sign, this requires a sign change to each term in that set.

$x(x + 2) - 2(x + 2) =$ ← take out common factors

$(x + 2)(x - 2)$ ← rewrite using common factors

4. $a^2 - 4 =$

5. $x^2 - 64 =$

6. $r^2 - 9 =$

7. $y^2 - 100 =$

8. $a^2 - 25b^2 =$

Practice

Use the list below to write the correct term for each **description** *on the line provided.*

binomial	**like terms**	**rational expression**
coefficient	**monomial**	**simplest form (of an expression)**
composite number	**polynomial**	
factor	**prime number**	**trinomial**

_____ 1. a number whose only factors are 1 and the number itself

_____ 2. the number part in front of an algebraic term signifying multiplication

_____ 3. a polynomial with exactly 2 terms

_____ 4. a polynomial with only 1 term

_____ 5. polynomials with exactly the same variable combinations.

_____ 6. a polynomial with exactly 3 terms

_____ 7. any rational expression with no variable in the denominator

_____ 8. a polynomial that contains no grouping symbols (except a fraction bar), and all like terms have been combined

_____ 9. one of the numbers multiplied to get a product

_____ 10. a fraction whose numerator and/or denominator are polynomials

_____ 11. a whole number that has more than two factors

Unit Review

Use the list below to identify each **polynomial**. *Write the word on the line provided.*

binomial	**monomial**	**trinomial**

_____ 1. $a + b + c$

_____ 2. $8xy^3z^2$

_____ 3. $4a^2 - b$

Write each **expression** *in* **simplest form**. *Show* **essential steps**.

4. $3a + [5a - (2a - b)] =$

5. $(x^3 - 4x^2y + 5xy^2 + 4y^3) - (-2x^3 + x^2y + 6xy^2 - 5y^3) =$

6. $8 - [2x - (3 + 5x) + 4] =$

7. $(3x^2)(-6x)^2 =$

8. $(-a)^2(4a^2)(-2a)^3 =$

9. $(5x)^2(-2x^2y^2)(4x) =$

10. $\dfrac{-16a^2b^5c^4}{4a^2bc^3} =$

11. $\dfrac{a^5b^2c^4}{a^5bc^3} =$

12. $\dfrac{(x-3)^4}{(x-3)^2} =$

13. $(x+5)^2 =$

14. $(a+2)(a-6) =$

15. $(3g-7)(2g+5) =$

16. $(a+5)(a^2-4a+9) =$

Factor *the following completely. Show* **essential steps**.

17. -32 =

18. $6x - 18 =$

19. $4m^3 - 16m^2 + 12m =$

20. $4(a - 2) - x(a - 2) =$

21. $3x^2 - 8x + 5 =$

22. $15x^2 - 16x + 4 =$

23. $y^2 + 10y + 25 =$

24. $x^2 - 4x + 4 =$

25. $x^2 - 36 =$

Unit 3: Making Sense of Rational Expressions

This unit emphasizes performing mathematical operations on rational expressions and using these operations to solve equations and inequalities.

Unit Focus

Number Sense, Concepts, and Operations

- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Algebraic Thinking

- Describe, analyze and generalize relationships, patterns and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)
- Determine the impact when changing parameters of given functions. (MA.D.1.4.2)

Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

canceling ..... dividing a numerator and a denominator by a common factor to write a fraction in lowest terms or before multiplying fractions
Example:
$\frac{15}{24} = \frac{{}^{1}\cancel{3} \bullet 5}{2 \bullet 2 \bullet 2 \bullet \cancel{3}_{1}} = \frac{5}{8}$

common denominator ..... a common multiple of two or more denominators
Example: A common denominator for $\frac{1}{4}$ and $\frac{5}{6}$ is 12.

common factor ..... a number that is a factor of two or more numbers
Example: 2 is a common factor of 6 and 12.

common multiple ..... a number that is a multiple of two or more numbers
Example: 18 is a common multiple of 3, 6, and 9.

cross multiplication ..... a method for solving and checking proportions; a method for finding a missing numerator or denominator in equivalent fractions or ratios by making the cross products equal

Example: To solve this proportion:

$$\frac{n}{9} = \frac{8}{12}$$

$$12 \text{ x } n = 9 \text{ x } 8$$

$$12n = 72$$

$$n = \frac{72}{12}$$

$$n = 6$$

Solution:

$$\frac{6}{9} = \frac{8}{12}$$

decimal number ..... any number written with a decimal point in the number

Example: A decimal number falls between two whole numbers, such as 1.5 falls between 1 and 2. Decimal numbers smaller than 1 are sometimes called decimal fractions, such as five-tenths is written 0.5.

denominator ..... the bottom number of a fraction, indicating the number of equal parts a whole was divided into

Example: In the fraction $\frac{2}{3}$ the denominator is 3, meaning the whole was divided into 3 equal parts.

difference ..... a number that is the result of subtraction

Example: In $16 - 9 = 7$, 7 is the difference.

distributive property ..... the product of a number and the sum or difference of two numbers is equal to the sum or difference of the two products
Example: $x(a + b) = ax + bx$

equation ..... a mathematical sentence in which two expressions are connected by an equality symbol
Example: $2x = 10$

equivalent (forms of a number) ..... the same number expressed in different forms
Example: $\frac{3}{4}$, 0.75, and 75%

expression ..... a collection of numbers, symbols, and/or operation signs that stands for a number
Example: $4r^2$; $3x + 2y$; $\sqrt{25}$
Expressions do *not* contain equality (=) or inequality (<, >, ≤, ≥, or ≠) symbols.

factor ..... a number or expression that divides evenly into another number; one of the numbers multiplied to get a product
Example: 1, 2, 4, 5, 10, and 20 are factors of 20 and $(x + 1)$ is one of the factors of $(x^2 - 1)$.

factoring ..... expressing a polynomial expression as the product of monomials and polynomials
Example: $x^2 - 5x + 4 = 0$
$(x - 4)(x - 1) = 0$

fraction ..... any part of a whole
Example: One-half written in fractional form is $\frac{1}{2}$.

inequality ..... a sentence that states one expression is greater than (>), greater than or equal to (≥), less than (<), less than or equal to (≤), or not equal to (≠) another expression
Example: $a \neq 5$ or $x < 7$ or $2y + 3 \geq 11$

integers ..... the numbers in the set
{… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

inverse operation ..... an action that undoes a previously applied action
Example: Subtraction is the inverse operation of addition.

irrational number ..... a real number that cannot be expressed as a ratio of two integers
Example: $\sqrt{2}$

least common denominator (LCD) ..... the smallest common multiple of the denominators of two or more fractions
Example: For $\frac{3}{4}$ and $\frac{1}{6}$, 12 is the least common denominator.

least common multiple (LCM) ..... the smallest of the common multiples of two or more numbers
Example: For 4 and 6, 12 is the least common multiple.

like terms ..... terms that have the same variables and the same corresponding exponents
Example: In $5x^2 + 3x^2 + 6$, $5x^2$ and $3x^2$ are like terms.

minimum ..... the smallest amount or number allowed or possible

multiplicative identity ..... the number one (1); the product of a number and the multiplicative identity is the number itself
Example: $5 \times 1 = 5$

multiplicative property of -1 ..... the product of any number and -1 is the opposite or additive inverse of the number
Example: $-1(a) = -a$ and $a(-1) = -a$

negative numbers ..... numbers less than zero

numerator ..... the top number of a fraction, indicating the number of equal parts being considered
Example: In the fraction $\frac{2}{3}$, the numerator is 2.

order of operations ..... the order of performing computations in parentheses first, then exponents or powers, followed by multiplication and/or division (as read from left to right), then addition and/or subtraction (as read from left to right); also called *algebraic order of operations*
Example:

$$\begin{aligned} 5 + (12 - 2) \div 2 - 3 \times 2 &= \\ 5 + 10 \div 2 - 3 \times 2 &= \\ 5 + 5 - 6 &= \\ 10 - 6 &= \\ 4 & \end{aligned}$$

polynomial ..... a monomial or sum of monomials; any rational expression with no variable in the denominator
Examples: $x^3 + 4x^2 - x + 8$ $\quad 5mp^2$
$-7x^2y^2 + 2x^2 + 3$

positive numbers ..... numbers greater than zero

product ..... the result of multiplying numbers together
Example: In 6 x 8 = 48,
48 is the product.

quotient ..... the result of dividing two numbers
Example: In 42 ÷ 7 = 6,
6 is the quotient.

ratio ..... the comparison of two quantities
Example: The ratio of a and b is a:b or $\frac{a}{b}$,
where $b \neq 0$.

rational expression ..... a fraction whose numerator and/or denominator are polynomials
Examples: $\frac{x}{8}$ $\quad \frac{5}{x+2}$ $\quad \frac{4x^2+1}{x^2+1}$

rational number ..... a real number that can be expressed as a ratio of two integers

real numbers ..... the set of all rational and irrational numbers

reciprocals ..... two numbers whose product is 1; also called *multiplicative inverses*
Example: Since $\frac{3}{4} \times \frac{4}{3} = 1$, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.

simplest form (of a fraction) ..... a fraction whose numerator and denominator have no common factor greater than 1
Example: The simplest form of $\frac{3}{6}$ is $\frac{1}{2}$.

simplify an expression ..... to perform as many of the indicated operations as possible

solution ..... any value for a variable that makes an equation or inequality a true statement
Example: In $y = 8 + 9$
$y = 17$ 17 is the solution.

substitute ..... to replace a variable with a numeral
Example: $8(a) + 3$
$8(5) + 3$

sum ..... the result of adding numbers together
Example: In $6 + 8 = 14$,
14 is the sum.

term ..... a number, variable, product, or quotient in an expression
Example: In the expression $4x^2 + 3x + x$, $4x^2$, $3x$, and x are terms.

variable ..... any symbol, usually a letter, which could represent a number

Unit 3: Making Sense of Rational Numbers

Introduction

Algebra students must be able to add, subtract, multiply, divide, and simplify rational expressions efficiently. These skills become more important as you become more involved in using mathematics. As a liberal arts mathematics student, you will have the opportunity to reacquaint yourself with the topics and methods you will need for future mathematical success.

Lesson One Purpose

- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Describe, analyze and generalize relationships, patterns and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)

Simplifying Rational Expressions

An **expression** is a collection of numbers, symbols, and/or operation signs that stand for a number. A **fraction**, or any part of a whole, is an *expression* that represents a **quotient**—the result of dividing two numbers. The same *fraction* may be expressed in many different ways.

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{5}{10}$$

If the **numerator** (top number) and the **denominator** (bottom number) are both **polynomials**, then we call the fraction a **rational expression**. A *rational expression* is a fraction whose *numerator* and/or *denominator* are *polynomials*. The fractions below are all rational expressions.

$$\frac{x}{x+y} \qquad \frac{a^2-2a+1}{a} \qquad \frac{1}{y^2+4} \qquad \frac{a}{b-3}$$

When the **variables** or any symbols which could represent numbers (usually letters) are replaced, the result is a numerator and a denominator that are **real numbers**. In this case, we say the entire *expression* is a *real number*. Real numbers are all **rational numbers** and **irrational numbers**—real numbers that *cannot* be expressed as a **ratio** of two **integers**. Of course, there is an exception…

when a denominator is equal to 0, we say the fraction is *undefined*.

Note: In this unit, we will agree that *no* denominator equals 0.

Fractions have some interesting properties. Let's examine them.

- If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. $\frac{4}{8} = \frac{6}{12}$ therefore $4 \bullet 12 = 8 \bullet 6$

  $\frac{a}{b} = \frac{c}{d}$

  $a \bullet d = b \bullet c$

  $ad = bc$

  In other words, if two fractions are equal, then the **products** are equal when you **cross multiply**.

  $\frac{4}{8} = \frac{6}{12}$

  $4 \bullet 12 = 8 \bullet 6$

  $48 = 48$

- $\frac{a}{b} = \frac{ac}{bc}$ $\frac{4}{7} = \frac{4 \bullet 3}{7 \bullet 3}$ therefore $\frac{4}{7} = \frac{12}{21}$

  Simply stated, if you *multiply* both the numerator and the denominator by the *same* number, the new fraction will be **equivalent** to the original fraction.

- $\frac{ac}{bc} = \frac{a}{b}$ $\frac{9}{21} = \frac{9 \div 3}{21 \div 3}$ therefore $\frac{9}{21} = \frac{3}{7}$

  Stated in words, if you *divide* both the numerator and the denominator by the *same* number, the new fraction will be *equivalent* to the original fraction. The same rules are true for **simplifying** rational expressions by performing as many indicated operations as possible. Many times, however, it is necessary to **factor** and find numbers or expressions that divide the numerator or the denominator, or both, so that the **common factors** become easier to see. Look at the following example:

  $$\frac{3x + 3y}{3} = \frac{{}^{1}\cancel{3}(x + y)}{\cancel{3}_{1}} = x + y$$

  Notice that by **factoring** a 3 out of the numerator, we can divide (or **cancel**) the 3s, leaving $x + y$ as the final result.

Before we move on, do the practice on the following pages.

Practice

Simplify *each* **expression**. *Refer to* **properties** *and* **examples** *on the previous pages as needed. Show* **essential steps**.

1. $\dfrac{4x-4}{x-1} =$

2. $\dfrac{4m-2}{2m-1} =$

3. $\dfrac{6x-3y}{3} =$

Example:

$$\frac{4x-6}{6} = \frac{{}^{1}\cancel{2}(2x-3)}{{}_{3}\cancel{6}} = \frac{1(2x-3)}{3} = \frac{2x-3}{3}$$

4. $\dfrac{5a-10}{15} =$

5. $\dfrac{2y-8}{4} =$

6. $\dfrac{3m+6n}{3} =$

7. $\dfrac{14r^3s^4+28rs^2-7rs}{7r^2s^2} =$

Practice

Simplify *each* **expression**. *Refer to* **properties** *and* **examples** *on the previous pages as needed. Show* **essential steps**.

Example:

$$\frac{2x^2 - 8}{x + 2} = \frac{2(x^2 - 4)}{x + 2} = \frac{2(x - 2)\cancel{(x + 2)}^{1}}{\cancel{x + 2}_{1}} = 2(x - 2)$$

Note: In the above example, notice the following:

→ after we factored 2 from the numerator

→ we were left with $x^2 - 4$

→ which can be factored into $(x + 2)(x - 2)$

→ then the $(x + 2)$ is cancelled

→ leaving $2(x - 2)$ as the final answer.

1. $\dfrac{3y^2 - 27}{y - 3} =$

2. $\dfrac{a - b}{a^2 - b^2} =$

3. $\dfrac{a-b}{b^2-a^2} =$

4. $\dfrac{9x^2+3}{6x^2+2} =$

5. $\dfrac{9x^2+3}{6x+3} =$

Additional Factoring

Look carefully at numbers 2-5 in the previous practice. What do you notice about them?

Alert! You cannot cancel individual **terms** (numbers, variables, products, or quotients in an expression)—you can only cancel factors (numbers or expressions that exactly divide another number)!

$$\frac{2x+4}{4} \neq \frac{2x}{4} \qquad \frac{3x+6}{3} \neq \frac{x+6}{3} \qquad \frac{9x^2+3}{6x+3} \neq \frac{9x^2x}{6x}$$

Look at how simplifying these expressions was taken a step further. Notice that additional factoring was necessary.

Example

$$\frac{x^2+5x+6}{x+3} = \frac{{}^{1}\cancel{(x+3)}(x+2)}{{}_{1}\cancel{x+3}} = (x+2) = x+2$$

Look at the denominator above. It is one of the factors of the numerator. Often, you can use the problem for hints as you begin to factor.

Practice

Factor *each of these and then* **simplify**. *Look for* **hints** *within the problem. Refer to the previous page as necessary. Show* **essential steps**.

1. $\dfrac{a^2 - 3a + 2}{a - 2} =$

2. $\dfrac{b^2 - 2b - 3}{b - 3} =$

Sometimes, it is necessary to **factor both** *the numerator and denominator. Examine the example below, then* **simplify** *each of the following* **expressions**.

Example:

$$\frac{x^2 - 4}{x^2 + x - 6} = \frac{(x + 2)\cancel{(x - 2)}^{1}}{(x + 3)\cancel{(x - 2)}_{1}} = \frac{(x + 2)}{(x + 3)} = \frac{x + 2}{x + 3}$$

Note: The x's do **not** cancel.

3. $\dfrac{2r^2 + r - 6}{r^2 + r - 2} =$

4. $\dfrac{x^2 + x - 2}{x^2 - 1} =$

Practice

Simplify *each* **expression**. *Show* **essential steps**.

1. $\dfrac{5b-10}{b-2} =$

2. $\dfrac{6a-9}{10a-15} =$

3. $\dfrac{9x+3}{9} =$

4. $\dfrac{6b+9}{12} =$

5. $\dfrac{3a^2b + 6ab - 9b^2}{3b} =$

6. $\dfrac{x^2 - 16}{x + 4} =$

7. $\dfrac{2a - b}{b^2 - 4a^2} =$

8. $\dfrac{6x^2 + 2}{9x^2 + 3} =$

Practice

Factor *each of these* **expressions** *and then* **simplify**. *Show* **essential steps**.

1. $\dfrac{y^2 + 5y - 14}{y + 2} =$

2. $\dfrac{a^2 - 5a + 4}{a - 4} =$

3. $\dfrac{6m^2 - m - 1}{2m^2 + 9m - 5} =$

4. $\dfrac{4x^2 - 9}{2x^2 + x - 6} =$

Practice

Use the list below to write the correct term for each definition on the line provided.

denominator	**numerator**	**rational expression**
expression	**polynomial**	**real numbers**
fraction	**quotient**	**variable**

_____ 1. a collection of numbers, symbols, and/or operation signs that stands for a number

_____ 2. the top number of a fraction, indicating the number of equal parts being considered

_____ 3. the bottom number of a fraction, indicating the number of equal parts a whole was divided into

_____ 4. the set of all rational and irrational numbers

_____ 5. any part of a whole

_____ 6. a fraction whose numerator and/or denominator are polynomials

_____ 7. any symbol, usually a letter, which could represent a number

_____ 8. a monomial or sum of monomials; any rational expression with no variable in the denominator

_____ 9. the result of dividing two numbers

Practice

Use the list below to complete the following statements.

canceling	**integers**
cross multiplication	**product**
equivalent	**simplify an expression**
factor	**terms**

1. If you *multiply* both the numerator and the denominator by the *same* number, the new fraction will be _____ because it is the same number expressed in a different form.

2. _____ are the numbers in the set {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}.

3. If you divide a numerator and a denominator by a common factor to write a fraction in lowest terms or before multiplying fractions, you are _____ .

4. To _____ , you need to perform as many of the indicated operations as possible.

5. Numbers, variables, products, or quotients in an expression are called _____ .

6. A _____ is a number or expression that divides evenly into another number.

7. When you multiply numbers together, the result is called the _____ .

8. To find a missing numerator or denominator in equivalent fractions or ratios, you can use a method called _____ and make the cross products equal.

Lesson Two Purpose

- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Describe, analyze and generalize relationships, patterns and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)

Addition and Subtraction of Rational Expressions

In order to add and subtract rational expressions in fraction form, it is necessary for the fractions to have a **common denominator** (the same bottom number). We find those *common denominators* in the same way we did with simple fractions. The process requires careful attention.

- When we add $\frac{3}{7} + \frac{5}{8}$, we find a common denominator by multiplying 7 and 8.
- Then we change each fraction to an equivalent fraction whose denominator is 56.

$$\frac{3 \bullet 8}{7 \bullet 8} = \frac{24}{56} \qquad \text{and} \qquad \frac{5 \bullet 7}{8 \bullet 7} = \frac{35}{56}$$

- Next we add $\frac{24}{56} + \frac{35}{56} = \frac{59}{56}$.

Finding the Least Common Multiple (LCM)

By multiplying the denominators of the terms we intend to add or subtract, we can always find a common denominator. However, it is often to our advantage to find the **least common denominator (LCD)**, which is also the **least common multiple (LCM)**. The *LCD* or *LCM* is the smallest of the **common multiples** of two or more numbers. This makes simplifying the result easier. Look at the example below.

Let's look at finding the LCM of 36, 27, and 15.

1. Factor each of the denominators and examine the results.

   $36 = 2 \bullet 2 \bullet 3 \bullet 3$ ← The new denominator must contain at least **two 2s** and **two 3s**.

   $27 = 3 \bullet 3 \bullet 3$ ← The new denominator must contain at least **three 3s**.

   $45 = 3 \bullet 3 \bullet 5$ ← The new denominator must contain at least **two 3s** and **one 5**.

2. Find the **minimum** combination of factors that is described by the combination of all the statements above—two 2s, three 3s, and one 5.

$$\text{LCM} = \underbrace{2 \bullet 2}_{\text{two 2s}} \bullet \underbrace{3 \bullet 3 \bullet 3}_{\text{three 3s}} \bullet \underbrace{5}_{\text{one 5}} = 540$$

3. Convert the *terms* to equivalent fractions using the new common denominator and then proceed to add or subtract.

$$\frac{5}{36} = \frac{75}{540};\ \frac{8}{27} = \frac{160}{540};\ \frac{4}{15} = \frac{144}{540} \quad \rightarrow \quad \frac{75}{540} + \frac{160}{540} - \frac{144}{540} = \frac{91}{540}$$

Now, let's look at an algebraic example.

$$\frac{y}{y^2 - 9} - \frac{1}{y^2 - 4y - 21} =$$

1. Factor each denominator and examine the results.

   $y^2 - 9 = (y + 3)(y - 3)$ ← The new denominator must contain $(y + 3)$ and $(y - 3)$.

   $y^2 - 4y - 21 = (y - 7)(y + 3)$ ← The new denominator must contain $(y - 7)$ and $(y + 3)$.

2. Find the *minimum* combination of factors.

   $\text{LCM} = (y + 3)(y - 3)(y - 7)$

3. Convert each fraction to an equivalent fraction using the new common denominator and proceed to subtract.

$$\frac{y(y-7)}{(y+3)(y-3)(y-7)} - \frac{1(y-3)}{(y+3)(y-3)(y-7)} =$$

$$\frac{y^2 - 7y - y + 3}{(y+3)(y-3)(y-7)} =$$

$$\frac{y^2 - 8y + 3}{(y+3)(y-3)(y-7)}$$

notice how the minus sign between the fractions *distributes* to make $-y + 3$ in the numerator (**distributive property**)

Hint: Always check to see if the numerator can be factored and then reduce, if possible.

Practice

Write each **sum** *or* **difference** *as a single fraction in* **lowest terms**. *Show* **essential steps**.

1. $\frac{a}{7} + \frac{2a}{7} - \frac{5}{7} =$

2. $\frac{x-2}{2y} + \frac{x}{2y} =$

3. $\frac{x+1}{5} + \frac{x-1}{5} =$

4. $\frac{x+1}{5} - \frac{x+1}{5} =$

5. $\frac{5}{6} + \frac{y}{4} =$

6. $\frac{2}{x+2} - \frac{3}{x+3} =$

Practice

Write each **sum** *or* **difference** *as a single fraction in* **lowest terms**. *Show* **essential steps**.

Example: $\dfrac{5}{b^2-9} - \dfrac{1}{b-3} = \dfrac{5}{(b+3)(b-3)} - \dfrac{1}{b-3} =$

$$\frac{5}{(b+3)(b-3)} - \frac{1(b+3)}{(b+3)(b-3)} =$$

$$\frac{5-1(b+3)}{(b+3)(b-3)} =$$

$$\frac{5-b-3}{(b+3)(b-3)} =$$

$$\frac{2-b}{(b+3)(b-3)} =$$

1. $\dfrac{1}{2z+1} + \dfrac{3}{z-2} =$

2. $\dfrac{r}{r^2-16} + \dfrac{r+1}{r^2-5r+4} =$

3. $\dfrac{8}{a^2 - 4} - \dfrac{2}{a^2 - 5a + 6} =$

4. $m + \dfrac{1}{m - 1} - \dfrac{1}{(m - 1)^2} =$

Practice

Write each **sum** *or* **difference** *as a single fraction in* **lowest terms**. *Show* **essential steps**.

1. $\frac{x}{3} - \frac{3y}{3} + \frac{4z}{3} =$

2. $\frac{x-2}{2y} - \frac{x}{2y} =$

3. $\frac{x+1}{5} - \frac{x-1}{5} =$

4. $\dfrac{2}{2a-4b} - \dfrac{b-2}{2a-4b} + \dfrac{7b}{2a-4b} =$

5. $\dfrac{x+3}{4} + \dfrac{5-x}{10} =$

6. $\dfrac{5}{2m-6} - \dfrac{3}{m-3} =$

Practice

Write each **sum** *or* **difference** *as a single fraction in* **lowest terms**. *Show* **essential steps**.

1. $\dfrac{2}{x^2 - x - 2} - \dfrac{2}{x^2 + 2x + 1} =$

2. $\dfrac{1}{b^2 - 1} - \dfrac{1}{b^2 + 2b + 1} =$

3. $\dfrac{3x}{x^2 + 3x - 10} - \dfrac{2x}{x^2 + x - 6} =$

4. $\dfrac{1}{x^2 - 7x + 12} + \dfrac{2}{x^2 - 5x + 6} - \dfrac{3}{x^2 - 6x + 8} =$

Lesson Three Purpose

- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Describe, analyze and generalize relationships, patterns and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)

Multiplication and Division of Rational Expressions

To multiply fractions, you learned to multiply the numerators together, then multiply the denominators together, and then reduce, if possible.

$$\frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$$

We use this same process with rational expressions.

$$\frac{4}{5x} \times \frac{11x}{13} = \frac{44x}{65x} = \frac{44}{65}$$

Sometimes it is simpler to reduce or cancel *common factors* before multiplying.

$$\frac{4}{5x} \times \frac{11x}{13} = \frac{4}{5\cancel{x}} \times \frac{11\cancel{x}}{13} = \frac{44}{65}$$

When we need to divide fractions, we invert (flip over) the factor to the right of the division symbol and then multiply.

invert

$$\frac{2x^2}{3y} \div \frac{4x}{5y^3} = \frac{2x^2}{3y} \bullet \frac{5y^3}{4x} = \frac{10x^2y^3}{12xy} = \frac{5xy^2}{6}$$

Pay careful attention to *negative* signs in the factors.

Decide before you multiply whether the answer will be positive *or* negative.

- If the number of negative factors is *even*, the result will be *positive*.
- If the number of negative factors is *odd*, the answer will be *negative*.

Remember: In this unit, we agreed that *no* denominator equals 0.

Practice

Write each **product** *as a single fraction in* **simplest terms**. *Show* **essential steps**.

1. $\frac{6x^3}{3} \bullet \frac{4b}{2x} =$

2. $\frac{14a^3b}{3b} \bullet \frac{-6}{7ab} =$

3. $\frac{-12ab^2}{5bc} \bullet \frac{10b^2c}{6ab} =$

Practice

Write each **product** *as a single fraction in* **simplest terms**. *Show* **essential steps**.

Example:

$$\frac{4a^2-1}{a^2-4} \bullet \frac{a+2}{4a+2} =$$

$$\frac{\cancel{(2a+1)}(2a-1)}{\cancel{(a+2)}(a-2)} \bullet \frac{\cancel{a+2}}{2\cancel{(2a+1)}} =$$

$$\frac{(2a-1)}{2(a-2)} =$$

$$\frac{2a-1}{2(a-2)}$$

1. $\dfrac{5x+25}{4x} \bullet \dfrac{2x}{3x+15} =$

2. $\dfrac{y^2-y-2}{y^2+4y+3} \bullet \dfrac{y^2-4y-5}{y^2-3y-10} =$

Hint: If you have trouble factoring, review the examples and processes on pages 180-182.

3. $\dfrac{2a^2 - a - 6}{3a^2 - 4a + 1} \bullet \dfrac{3a^2 + 7a + 2}{2a^2 + 7a + 6} =$

4. $\dfrac{3x^2 - 3x}{5} \bullet \dfrac{x^2 - 9x - 10}{6x - 60} \bullet \dfrac{4}{1 - x^2} =$

Practice

Write each **quotient** *as a single fraction in* **simplest terms**. *Show* **essential steps**.

Remember: Invert and then multiply!

1. $\frac{9ab}{x} \div \frac{3a}{2x^2} =$

2. $\frac{x^2 - x - 6}{x^2 - 2x - 15} \div \frac{x^2 - 4}{x^2 - 6x + 5} =$

3. $\frac{10a^2 - 13a - 3}{2a^2 - a - 3} \div \frac{5a^2 - 9a + -2}{3a^2 + 2a - 1} =$

4. $\frac{9r^2 + 3r - 2}{12r^2 + 5r - 2} \div \frac{9r^2 - 6r + 1}{8r^2 + 10r - 3} =$

Practice

Write each **product** *as a single fraction in* **simplest terms**. *Show* **essential steps**.

1. $\frac{4a^3}{3} \bullet \frac{6b}{2a} =$

2. $\frac{-18ab^2}{5bc} \bullet \frac{15b^3c}{6ab} =$

3. $\frac{24a^3b}{3b} \bullet \frac{-9}{6ab} =$

4. $\dfrac{9b^2 - 25}{2b - 2} \bullet \dfrac{b^2 - 1}{6b - 10} =$

5. $\dfrac{x^2 - x - 20}{x^2 + 7x + 12} \bullet \dfrac{x + 3}{x - 5} =$

6. $\dfrac{7x + 14}{14x - 28} \bullet \dfrac{4 - 2x}{x + 2} \bullet \dfrac{x + 3}{x + 1} =$

Hint: $a - b = -(b - a)$

Practice

Write each **quotient** *as a single fraction in* **simplest terms**. *Show* **essential steps**.

1. $\dfrac{28x^2y^3}{10a^2} \div \dfrac{21x^3y}{5a} =$

2. $\dfrac{4x-8}{3} \div \dfrac{-(6x-12)}{9} =$

3. $\dfrac{6a^3b}{4x} \div \dfrac{3a}{2x^3} =$

4. $\dfrac{r^2+2r-15}{r^2+3r-10} \div \dfrac{r^2-9}{r^2-9r+14} =$

5. $\dfrac{y^2+y-2}{y^2+2y-3} \div \dfrac{y^2+7y+10}{y^2-2y-15} =$

Lesson Four Purpose

- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Describe, analyze and generalize relationships, patterns and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)
- Determine the impact when changing parameters of given functions. (MA.D.1.4.2)

Solving Equations

An **equation** is a mathematical sentence that uses an equal sign to show that two quantities are equal. An *equation* equates one expression to another.

$3x - 7 = 8$ is an example of a equation.

You may be able to solve this problem mentally, without using paper and pencil.

$3x - 7 = 8$

The problem reads—3 times what number minus 7 equals 8?

Think: $3 \bullet 4 = 12$
$12 - 7 = 5$ ← too small

Think: $3 \bullet 5 = 15$
$15 - 7 = 8$ ← That's it!

$3x - 7 = 8$
$3(5) - 7 = 8$

Practice

Solve each of the following **mentally**, *writing only the answer.*

1. $4y + 6 = 22$

   $y =$

2. $2a - 4 = 10$

   $a =$

3. $5x - 15 = -20$

   $x =$

4. $-7b + 6 = -22$

   $b =$

✓ **Check yourself:** Add all your answers for problems 1-4. Did you get a *sum* of 14? If not, correct your work before continuing.

Step-by-Step Process for Solving Equations

A problem like $\frac{x+12}{5} = -2(x-10)$ is a bit more challenging. You could use a guess and check process, but that would take more time, especially when answers involve **decimals** or fractions.

So, as problems become more difficult, you can see that it is important to have a process in mind and to write down the steps as you go.

Unfortunately, there is *no* exact process for solving equations. Every rule has an exception. That is why practice is necessary and keeping a written record of the steps you have used is extremely helpful.

Example 1

Let's look at a step-by-step process for solving the problem above.

$$\frac{x+12}{5} = -2(x-10)$$ ← Step 1: Copy the problem **carefully**!

$$\frac{x+12}{5} = -2x + 20$$ ← Step 2: Simplify each side of the equation as needed by *distributing* the 2.

$$\left(\frac{x+12}{\cancel{5}}\right) \bullet \cancel{5} = (-2x + 20) \bullet 5$$ ← Step 3: Multiply both sides of the equation by 5 to "undo" the division by 5, which eliminates the fraction.

$$x + 12 = -10x + 100$$ ← Step 4: Simplify by distributing the 5.

$$+10x + 1x + 12 = -10x + 10x + 100$$
$$11x + 12 = 100$$
← Step 5: Add 10*x* to both sides.

$$11x + 12 - 12 = 100 - 12$$
$$11x = 88$$
← Step 6: Subtract 12 from both sides.

$$11x \div 11 = 88 \div 11$$
$$x = 8$$
← Step 7: Divide both sides by 11.

$$\frac{x+12}{5} = -2(x-10)$$
$$\frac{8+12}{5} = -2(8) + 20$$
$$4 = -16 + 20$$
$$4 = 4$$
← Step 8: Check by replacing the variable in the original problem.

It checks!

Example 2

What if the original problem had been $5x + 12 = -2(x - 10)$? The process would have been different. Watch for differences.

$5x + 12 = -2(x - 10)$	← Step 1:	Copy the problem **carefully**!
$5x + 12 = -2x + 20$	← Step 2:	Simplify each side of the equation as needed by *distributing* the 2.
$5x + 12 - 12 = -2x + 20 - 12$ $5x = -2x + 8$	← Step 3:	Subtract 12 from both sides of the equation.
$5x + 2x = -2x + 2x + 8$ $7x = 8$	← Step 4:	Add $2x$ to both sides of the equation.
$7x \div 7 = 8 \div 7$ $x = \frac{8}{7}$	← Step 5:	Divide both sides by 7.
$5x + 12 = -2(x - 10)$ $5(\frac{8}{7}) + 12 = -2(\frac{8}{7}) + 20$ $\frac{40}{7} + 12 = \frac{-16}{7} + 20$ $5\frac{5}{7} + 12 = -2\frac{2}{7} + 20$ $17\frac{5}{7} = 17\frac{5}{7}$	← Step 6: ←	Check by replacing the variable in the original problem. *It checks!*

Did you notice that the steps were *not* always the same? The rules for solving equations change to fit the individual needs of each problem. You can see why it is a good idea to check your answers each time. You may need to do some steps in a different order than you originally thought.

Generally speaking the processes for solving equations are as follows.

- Simplify both sides of the equation as needed.
- "Undo" additions and subtractions.
- "Undo" multiplications and divisions.

You might notice that this seems to be the *opposite* of the **order of operations**. Typically, we "undo" in the *reverse* order from the original process.

Guidelines for Solving Equations

1. Use the **distributive property** to clear parentheses.
2. Combine **like terms**. We want to isolate the variable.
3. Undo addition or subtraction using **inverse operations**.
4. Undo multiplication or division using *inverse operations*.
5. Check by **substituting** the **solution** in the original equation.

SAM = **S**implify (steps 1 and 2) then
Add (or subtract)
Multiply (or divide)

Here are some additional examples.

Example 3

Solve:

$$\begin{aligned} 6y + 4(y + 2) &= 88 \\ 6y + 4y + 8 &= 88 && \leftarrow \text{use } \textit{distributive property} \\ 10y + 8 - 8 &= 88 - 8 && \leftarrow \text{combine } \textit{like terms} \text{ and undo addition by subtracting 8 from each side} \\ \frac{10y}{10} &= \frac{80}{10} && \leftarrow \text{undo multiplication by dividing by 10} \\ y &= 8 \end{aligned}$$

Check *solution* in the original equation:

$$\begin{aligned} 6y + 4(y + 2) &= 88 \\ 6(8) + 4(8 + 2) &= 88 \\ 48 + 4(10) &= 88 \\ 48 + 40 &= 88 \\ 88 &= 88 && \leftarrow \textit{It checks!} \end{aligned}$$

Example 4

Solve:

$$\begin{aligned} -\tfrac{1}{2}(x + 8) &= 10 \\ -\tfrac{1}{2}x - 4 &= 10 && \leftarrow \text{use distributive property} \\ -\tfrac{1}{2}x - 4 + 4 &= 10 + 4 && \leftarrow \text{undo subtraction by adding 4 to both sides} \\ -\tfrac{1}{2}x &= 14 \\ (-2)-\tfrac{1}{2}x &= 14(-2) && \leftarrow \text{isolate the variable by multiplying each side by the } \textbf{reciprocal} \text{ of } -\tfrac{1}{2} \\ x &= -28 \end{aligned}$$

Check solution in the original equation:

$$\begin{aligned} -\tfrac{1}{2}(x + 8) &= 10 \\ -\tfrac{1}{2}(-28 + 8) &= 10 \\ -\tfrac{1}{2}(-20) &= 10 \\ 10 &= 10 && \leftarrow \textit{It checks!} \end{aligned}$$

Example 5

Solve:

$$
\begin{aligned}
26 &= \tfrac{2}{3}(9x - 6) \\
26 &= \tfrac{2}{3}(9x) - \tfrac{2}{3}(6) && \leftarrow \text{use distributive property} \\
26 &= 6x - 4 \\
26 + 4 &= 6x - 4 + 4 && \leftarrow \text{undo subtraction by adding 4 to each side} \\
\tfrac{30}{6} &= \tfrac{6x}{6} && \leftarrow \text{undo multiplication by dividing each side by 6} \\
5 &= x
\end{aligned}
$$

Check solution in the original equation:

$$
\begin{aligned}
26 &= \tfrac{2}{3}(9x - 6) \\
26 &= \tfrac{2}{3}(9 \bullet 5 - 6) \\
26 &= \tfrac{2}{3}(39) \\
26 &= 26 && \leftarrow \textit{It checks!}
\end{aligned}
$$

Example 6

Solve:

$$
\begin{aligned}
x - (2x + 3) &= 4 \\
x - 1(2x + 3) &= 4 && \leftarrow \text{use the } \textbf{multiplicative property of -1} \\
x - 2x - 3 &= 4 && \leftarrow \text{use the } \textbf{multiplicative identity of 1} \\
& && \quad \text{and use the distributive property} \\
-1x - 3 &= 4 && \leftarrow \text{combine like terms} \\
-1x - 3 + 3 &= 4 + 3 && \leftarrow \text{undo subtraction} \\
\frac{-1x}{-1} &= \frac{7}{-1} && \leftarrow \text{undo multiplication} \\
x &= -7
\end{aligned}
$$

Examine the solution steps above. See the use of the *multiplicative property of -1* in front of the parentheses on line two.

line 1: $x - (2x + 3) = 4$
line 2: $x - \mathbf{1}(2x + 3) = 4$

Also notice the use of *multiplicative identity* on line three.

line 3: $\mathbf{1}x - 2x - 3 = 4$

The simple variable x was multiplied by 1 ($1 \bullet x$) to equal $1x$. The $1x$ helped to clarify the number of variables when combining like terms on line four.

Check solution in the original equation:

$$
\begin{aligned}
x - (2x + 3) &= 4 \\
-7 - (2 \bullet -7 + 3) &= 4 \\
-7 - (-11) &= 4 \\
4 &= 4 && \leftarrow \textit{It checks!}
\end{aligned}
$$

Practice

Solve *and* **check** *each* **equation**. *Use the examples on pages 200-205 for reference. Show* **essential steps**.

Hint: Find a step that looks similar to the problem you need help with and follow from that point.

Remember: To check your work, replace the variable in the original problem with the answer you found.

1. $3x - 7 = 17$

2. $4x + 20 = x - 4$

3. $\frac{x}{6} = 1.5$

4. $\frac{2x}{5} = 3.2$

5. $5(x - 4) = 20$

6. $5(4x - 7) = 0$

7. $8x - 2x = 42$

8. $5x - 3 = 2x + 18$

9. $-2x + 4 = -4x - 10$

Practice

Solve *and* **check** *each* **equation**. *Use the examples on pages 200-205 for reference. Show* **essential steps**.

1. $2(3x - 4) + 6 = 10$

2. $3(x - 7) - x = -9$

3. $\frac{2}{3}x = 1$

   Hint: $\frac{2}{3}x = \frac{2x}{3}$. Rewrite 1 as $\frac{1}{1}$ and cross multiply.

4. $\frac{-1}{2}x - \frac{3}{4} = 4$

5. $-3x = \frac{-33}{8}$

6. $\frac{-2}{x} = 8$

7. $-3x - \frac{3}{2} = \frac{11}{2}$

Practice

Solve *and* **check** *each* **equation**.

1. $-87 = 9 - 8x$

2. $4k + 3 = 3k + 1$

3. $5a + 9 = 64$

4. $\frac{b}{3} + 5 = -2$

5. $4x = -(9 - x)$

6. $\frac{5}{x} = -10$

7. $3x - 1 = -x + 19$

Practice

Solve *and* **check** *each* **equation**. **Reduce** *fractions to* **simplest form**.

1. $5x - 3 = 2x + 18$

2. $6x - (4x - 12) = 3x + 5$

3. $\frac{x}{6} = \frac{-24}{5}$

4. $4(x - 2) = -3(x + 5)$

5. $5(\frac{1}{3}x - 2) = 4$

6. $\frac{4}{x} + \frac{3}{2} = \frac{5}{8}$

7. $\frac{2}{9}x = \frac{1}{5}$

8. $\frac{-1}{2} + \frac{8x}{5} = \frac{-7}{8}$

Lesson Five Purpose

- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Describe, analyze and generalize relationships, patterns and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)
- Determine the impact when changing parameters of given functions. (MA.D.1.4.2)

Solving Inequalities

Inequalities are mathematical sentences that are not equal. Instead of using the equal symbol (=), we use the following with *inequalities*.

- greater than $>$
- less than $<$
- greater than or equal to $\geq$
- less than or equal to $\leq$
- not equal to $\neq$

Remember: The "*is greater than*" (>) or "*is less than*" (<) symbols always *point to the lesser number*.

For example:

$5 > 3$
$3 < 5$

We have been solving *equations* in this unit. When we solve inequalities, the procedures are the same except for one important difference.

When we multiply or divide both sides of an inequality by the same *negative number*, we *reverse* the direction of the inequality symbol.

Example

Solve by *dividing* by a **negative number** and *reversing* the inequality sign.

$$-3x < 6$$

$$\frac{-3x}{-3} > \frac{6}{-3}$$ ← divide each side by -3 and reverse the inequality symbol

$$x > -2$$

To check this solution, pick any number *greater than* -2 and substitute your choice into the original inequality. For instance, -1, 0, or 3, or 3,000 could be substituted into the original problem.

Check with different solutions of numbers *greater than* -2:

substitute -1

$$-3x < 6$$
$$-3(-1) < 6$$
$$3 < 6$$ ← *It checks!*

substitute 3

$$-3x < 6$$
$$-3(3) < 6$$
$$-9 < 6$$ ← *It checks!*

substitute 0

$$-3x < 6$$
$$-3(0) < 6$$
$$0 < 6$$ ← *It checks!*

substitute 3,000

$$-3x < 6$$
$$-3(3{,}000) < 6$$
$$-9{,}000 < 6$$ ← *It checks!*

Notice that -1, 0, 3, and 3,000 are all *greater than* -2 and each one *checks* as a solution.

Study the following examples.

Example

Solve by *multiplying* by a *negative number* and *reversing* the inequality sign.

$$-\frac{1}{3}y \geq 4$$

$$(-3)\,-\frac{1}{3}y \leq 4(-3)$$ ← multiply each side by -3 and reverse the inequality symbol

$$y \leq -12$$

Example

Solve by first adding, then *dividing* by a *negative number*, and *reversing* the inequality sign.

$$-3a - 4 > 2$$

$$-3a - 4 + 4 > 2 + 4$$ ← add 4 to each side

$$-3a > 6$$

$$\frac{-3a}{-3} < \frac{6}{-3}$$ ← divide each side by -3 and reverse the inequality symbol

$$a < -2$$

Example

Solve by first subtracting, then *multiplying* by a *negative number*, and *reversing* the inequality sign.

$$\frac{y}{-2} + 5 \leq 0$$

$$\frac{y}{-2} + 5 - 5 \leq 0 - 5$$ ← subtract 5 from each side

$$\frac{y}{-2} \leq -5$$

$$\frac{(-2)y}{-2} \geq (-5)(-2)$$ ← multiply each side by -2 and reverse the inequality symbol

$$y \geq 10$$

Example

Solve by first subtracting, then *multiplying* by a **positive number**. Do **not** *reverse* the inequality sign.

$$\frac{n}{2} + 5 \leq 2$$

$$\frac{n}{2} + 5 - 5 \leq 2 - 5 \leftarrow \text{subtract 5 from each side}$$

$$\frac{n}{2} \leq -3$$

$$\frac{(2)n}{2} \leq -3(2) \leftarrow \text{multiply each side by 2, } \textit{but}$$

$$n \leq -6$$

do *not* reverse the inequality symbol because we multiplied by a *positive number*

When multiplying or dividing both sides of an inequality by the same *positive number*, do *not* reverse the inequality symbol—leave it alone.

Example

Solve by first adding, then *dividing* by a positive number. Do **not** *reverse* the inequality sign.

$$7x - 3 > -24$$

$$7x - 3 + 3 > -24 + 3 \leftarrow \text{add 3 to each side}$$

$$7x > -21$$

$$\frac{7x}{7} > \frac{-21}{7} \leftarrow \text{divide each side by 7, but}$$

$$x > -3$$

do *not* reverse the inequality symbol because we divided by a positive number

Practice

Solve each **inequality** *on the following page. Use the examples below and pages 214-217 for reference. Show* **essential steps**.

Remember: Reverse the inequality symbol every time we multiply or divide both sides of the inequality by a negative number. See the example below.

Example:

$$7 - 3x > 13$$

$$7 - 7 - 3x > 13 - 7 \quad \leftarrow \text{subtract 7 from both sides}$$

$$-3x > 6$$

$$\frac{\cancel{-3}x}{\cancel{-3}} > \frac{\cancel{6}^{2}}{\cancel{-3}} \quad \leftarrow \text{divide both sides by -3 and reverse the inequality symbol}$$

$$x < -2$$

Notice in the example above that we first subtracted 7 from both sides of the sentence. Then we solved for x, we divided both sides by -3 and the > symbol became a < symbol.

Check your answer by choosing a number that fits your answer. Replace the variable in the original sentence with the chosen number. Check to see if it makes a *true* statement.

In the example above, choose a number that makes $x < -2$ a *true* statement. For example, let's try -3.

Now put -3 in place of the variable in the original problem and see what happens.

$$7 - 3x > 13 \quad \leftarrow \text{original sentence}$$

$$7 - 3(-3) > 13 \quad \leftarrow \text{replace } x \text{ with -3}$$

$$7 - (-9) > 13$$

$$7 + 9 > 13$$

$$16 > 13 \quad \leftarrow \text{This is a true statement, so the answer } (x < -2) \text{ is correct.}$$

See **directions** *and* **examples** *on previous page.*

1. $6x - 7 > 17$

2. $13x + 20 < x - 4$

3. $\frac{x}{5} \geq 1.5$

4. $\frac{2x}{5} > 4.8$

5. $5(x - 4) < 20$

6. $3(4x - 7) \geq 15$

7. $3(x - 7) - x > -9$

8. $\frac{-1}{2}x - \frac{3}{4} \leq 6$

9. $2x - 9 < -21$

10. $\frac{-12}{x} < 8$

11. $4(x - 7) - x > -7$

12. $\frac{2}{3}x > 10$

Practice

Solve each **inequality**. *Show* **essential steps**.

1. $5x - 3 \leq 12$

2. $2a + 7 \geq 5a - 5$

3. $\frac{2x}{5} > 2.4$

4. $5(x - 5) < 20$

5. $-2(x + 6) > 14$

6. $8x - 12x > 48$

7. $5x - 3 \geq 2x + 18$

8. $-2x + 4 < -4x - 12$

9. $2(3x - 4) + 6 \leq 16$

10. $-5x - \frac{3}{2} \geq \frac{11}{2}$

11. $-3x > \frac{-33}{7}$

12. $\frac{2}{3}x > 11$

Practice

Write **True** *if the statement is correct. Write* **False** *if the statement is* not *correct.*

_____ 1. An equation is a mathematical sentence that uses an equal sign to show that two quantities are equal.

_____ 2. A product is the result of dividing two numbers.

_____ 3. A quotient is the result of multiplying two numbers.

_____ 4. An expression is a collection of numbers, symbols, and/or operation signs that stand for a number.

_____ 5. To simplify an expression, perform as many indicated operations as possible.

_____ 6. A common multiple is a number that is a multiple of two or more numbers.

_____ 7. The smallest of the common multiples of two or more numbers is called the least common multiple (LCM).

_____ 8. A number that is the result of subtraction is called the sum.

_____ 9. A number that is the result of adding numbers together is called the difference.

_____ 10. To solve an inequality, you will have to reverse the inequality symbol every time you add or subtract both sides of the inequality by a negative number.

Unit Review

Simplify each **expression**.

1. $\dfrac{5x - 10}{x - 2} =$

2. $\dfrac{6x - 9y}{3} =$

3. $\dfrac{12a^2b^5 + 18a^3b^4 - 24a^4b^3}{-6a^2b^3} =$

4. $\dfrac{12x - 6}{10x - 5} =$

5. $\dfrac{x^2 - 4}{x^2 + x - 6} =$

6. $\dfrac{x^2 + 3x - 10}{x + 5} =$

Write each **sum** *or* **difference** *as a single fraction in* **lowest terms**.

7. $\dfrac{3a}{8} + \dfrac{a}{8} - \dfrac{6}{8} =$

8. $\dfrac{x + 3}{6} - \dfrac{x - 3}{6} =$

9. $\dfrac{x-2}{4} + \dfrac{x+2}{4} =$

10. $\dfrac{2}{3x+1} + \dfrac{5}{x-3} =$

11. $\dfrac{3}{a^2-9} - \dfrac{6}{a^2+a-6} =$

Write each **product** *or* **quotient** *as a single fraction in* **simplest terms**.

12. $\dfrac{21x^2y^3}{3xy} \bullet \dfrac{-9}{7xy^2} =$

13. $\dfrac{a}{a+4} \bullet \dfrac{3a+12}{6} =$

14. $\dfrac{-12}{x^2-x} \div \dfrac{4x-2}{x^2-1} =$

15. $\dfrac{x^2-x-20}{x^2+7x+12} \bullet \dfrac{x^2+9x+18}{x^2-7x+10} =$

Solve each **equation**.

16. $3(4x - 2) = 30$

17. $7x - 2(x + 3) = 19$

18. $5 - \frac{x}{2} = 12$

19. $28 + 6x = 23 + 8x$

Solve each **inequality**.

20. $5x + 4 \geq 20$

21. $16 - 4x < 20$

22. $5(x + 2) > 4x + 7$

Unit 4: How Radical Are You?

This unit focuses on simplifying radical expressions and performing operations involving radicals.

Unit Focus

Number Sense, Concepts, and Operations

- Add, subtract, multiply and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

coefficient ..... a numerical factor in a term of an algebraic expression
Example: In $8a$, the coefficient of a is 8.

conjugate ..... if $x = a + b$, then $a - b$ is the conjugate of x
Example: The expressions $(a + \sqrt{b})$ and $(a - \sqrt{b})$ are conjugates of each other.

decimal number ..... any number written with a decimal point in the number
Example: A decimal number falls between two whole numbers, such as 1.5 falls between 1 and 2. Decimal numbers smaller than 1 are sometimes called decimal fractions, such as five-tenths is written 0.5.

denominator ..... the bottom number of a fraction, indicating the number of equal parts a whole was divided into
Example: In the fraction $\frac{2}{3}$ the denominator is 3, meaning the whole was divided into 3 equal parts.

digit ..... any one of the 10 symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9

distributive property ..... the product of a number and the sum or difference of two numbers is equal to the sum or difference of the two products
Example: $x(a + b) = ax + bx$

expression ..... a collection of numbers, symbols, and/or operation signs that stands for a number
Example: $4r^2$; $3x + 2y$; $\sqrt{25}$
Expressions do *not* contain equality (=) or inequality (<, >, ≤, ≥, or ≠) symbols.

factor ..... a number or expression that divides evenly into another number
Example: 1, 2, 4, 5, 10, and 20 are factors of 20 and $(x + 1)$ is one of the factors of $(x^2 - 1)$.

FOIL method ..... a pattern used to multiply two binomials; multiply the first, outside, inside, and last terms:

F	First terms
O	Outside terms
I	Inside terms
L	Last terms.

Example:

2 Outside
1 First
$$(a + b)(x - y) = \overset{\text{F}}{ax} - \overset{\text{O}}{ay} + \overset{\text{I}}{bx} - \overset{\text{L}}{by}$$
3 Inside
4 Last

fraction ..... any part of a whole
Example: One-half written in fractional form is $\frac{1}{2}$.

irrational number ..... a real number that cannot be expressed as a ratio of two integers
Example: $\sqrt{2}$

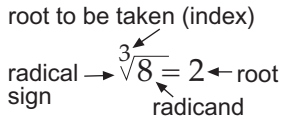
like terms ..... terms that have the same variables and the same corresponding exponents
Example: In $5x^2 + 3x^2 + 6$, $5x^2$ and $3x^2$ are like terms.

numerator ..... the top number of a fraction, indicating the number of equal parts being considered
Example: In the fraction $\frac{2}{3}$, the numerator is 2.

perfect square ..... a number whose square root is a whole number
Example: 25 is a perfect square because 5 x 5 = 25

product ..... the result of multiplying numbers together
Example: In 6 x 8 = 48,
48 is the product.

radical ..... an expression that has a root (square root, cube root, etc.)
Example: $\sqrt{25}$ is a radical
Any root can be specified by an index number, b, in the form $\sqrt[b]{a}$ (e.g., $\sqrt[3]{8}$).
A radical without an index number is understood to be a square root.



radical

radical expression ..... a numerical expression containing a radical sign
Examples: $\sqrt{25}$ $2\sqrt{25}$

radical sign ($\sqrt{\ }$) ..... the symbol ($\sqrt{\ }$) used before a number to show that the number is a *radicand*

rationalizing the denominator ..... a method used to remove or eliminate radicals from the denominator of a fraction

rational number ..... a real number that can be expressed as a ratio of two integers

simplest radical form ..... a number under the radical sign that is not a product with a perfect square factor
Example: $\sqrt{27} = \sqrt{9 \bullet 3} = \sqrt{9} \bullet \sqrt{3} = 3\sqrt{3}$

simplify an expression ..... to perform as many of the indicated operations as possible

square root ..... a positive real number that can be multiplied by itself to produce a given number
Example: The square root of 144 is 12 or $\sqrt{144} = 12$.

term ..... a number, variable, product, or quotient in an expression
Example: In the expression $4x^2 + 3x + x$, $4x^2$, $3x$, and x are terms.

variable ..... any symbol, usually a letter, which could represent a number

whole number ..... the numbers in the set {0, 1, 2, 3, 4, …}

Unit 4: How Radical Are You?

Introduction

We will see that radical expressions can be rewritten to conform to the mathematical definitions of simplest terms. We will then be able to perform the operations of addition, subtraction, multiplication and division on these reformatted expressions. We will also explore the effects of multiplying a radical expression by its conjugate.

Lesson One Purpose

- Add, subtract, multiply and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Simplifying Radical Expressions

A **radical expression** is any mathematical **expression** that contains a **square root** symbol. Look at the following examples:

$$\sqrt{5} \qquad \frac{\sqrt{6}}{3} \qquad \frac{3}{\sqrt{6}} \qquad \frac{7}{5+\sqrt{2}} \qquad \sqrt{36}$$

In an earlier unit, we discussed reformatting certain numbers to make them easier to work with. We also learned that mathematicians liked to have rules that make working with numbers uniform. If we all play by the same rules, we should all have the same outcome.

With this in mind, here are the two basic rules for working with *square roots*.

1. Never leave a **perfect square factor** under a **radical sign ($\sqrt{\ }$)**.

2. Never leave a *radical sign* in a **denominator**.

Important! Do **not** use your calculator with the *square roots*. It will change the numbers to **decimal** approximations. We are looking for exact answers.

Let's explore each of the rules…one at a time.

Rule One

First, let's review the idea of *perfect squares*. Perfect squares happen whenever you multiply a number times itself. For instance,

$3 \times 3 = 9$ $\quad$ $7 \times 7 = 49$ $\quad$ $9 \times 9 = 81$

9, 49, and 81 are all perfect squares.

It will be helpful to learn the chart below. You will be asked to use these numbers many times in this unit. The chart shows the perfect squares underneath the radical sign, then gives the square root of each perfect square.

Perfect Squares:
Square Root = Whole Number

$\sqrt{1}$	=	1
$\sqrt{4}$	=	2
$\sqrt{9}$	=	3
$\sqrt{16}$	=	4
$\sqrt{25}$	=	5
$\sqrt{36}$	=	6
$\sqrt{49}$	=	7
$\sqrt{64}$	=	8
$\sqrt{81}$	=	9
$\sqrt{100}$	=	10
$\sqrt{121}$	=	11
$\sqrt{144}$	=	12
$\sqrt{169}$	=	13
$\sqrt{196}$	=	14
$\sqrt{225}$	=	15
$\sqrt{256}$	=	16
$\sqrt{289}$	=	17
$\sqrt{324}$	=	18
$\sqrt{361}$	=	19
$\sqrt{400}$	=	20

Any time you see a perfect square under a square root symbol, simplify it by writing it as the square root.

Sometimes, perfect squares are hidden in an *expression* and we have to search for them. At first glance, $\sqrt{45}$ looks as if it is in **simplest radical form**. However, when we realize that 45 has a *factor* that is a perfect square, we can rewrite

$$\sqrt{45} = \sqrt{9} \bullet \sqrt{5}.$$

From the information in the chart, we know that 9 is a perfect square and that

$\sqrt{9} = 3$. Therefore,

$$\sqrt{45} = 3 \bullet \sqrt{5} \text{ or } 3\sqrt{5}.$$

Let's look at some examples.

$$\begin{aligned}\sqrt{18} &= \sqrt{9} \bullet \sqrt{2}\\ &= 3 \bullet \sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

$$\begin{aligned}\sqrt{20} &= \sqrt{4} \bullet \sqrt{5}\\ &= 2 \bullet \sqrt{5}\\ &= 2\sqrt{5}\end{aligned}$$

Now you try some in the following practices. Study the chart of perfect squares on page 240 before you start the practices.

Practice

Simplify *each* **radical expression**.

Remember: Never leave a perfect square factor under a radical sign.

1. $\sqrt{50} =$

2. $\sqrt{27} =$

3. $\sqrt{125} =$

4. $\sqrt{64} =$

5. $\sqrt{13} =$

6. $\sqrt{32} =$

7. $\sqrt{12} =$

8. $\sqrt{45} =$

9. $\sqrt{300} =$

10. $\sqrt{8} =$

Practice

Simplify *each* radical expression.

1. $\sqrt{48} =$

2. $2\sqrt{40} =$

3. $\sqrt{60} =$

4. $\sqrt{242} =$

5. $\sqrt{28} =$

6. $-\sqrt{200} =$

7. $-\sqrt{250} =$

8. $\sqrt{108} =$

9. $\sqrt{405} =$

10. $5\sqrt{90} =$

Rule Two

Now it's time to work on that second rule…never leave a square root in the *denominator*.

If a **fraction** has a denominator that is a perfect square root, just rewrite the *fraction* using that square root. Let's look at examples.

$$\frac{2}{\sqrt{36}} = \frac{2}{6} = \frac{1}{3} \qquad \frac{4}{\sqrt{81}} = \frac{4}{9}$$

Many times, however, that denominator will *not* be a perfect square root. In those cases, we have to reformat the denominator so that it is a perfect square root. This is called **rationalizing the denominator** or the bottom number of the fraction. To do this, we make it into a **rational number** by using a method to eliminate **radicals** from the denominator of a fraction. Remember, we aren't concerned about what may happen to the format of the **numerator**…just the denominator.

To reformat an **irrational** denominator (one with a square root in it), we find a number to multiply it by that will produce a perfect square root.

Follow the explanation of this example carefully.

$\frac{2}{\sqrt{7}}$ ← Yikes! This denominator is irrational! I need to *rationalize* it.

$\frac{2}{\sqrt{7}} \bullet \frac{\sqrt{7}}{\sqrt{7}}$ ← Look what will happen if I multiply the denominator by itself. (Since, $\frac{\sqrt{7}}{\sqrt{7}} = 1$, I have *not* changed the value of the original fraction.)

$\frac{2}{\sqrt{7}} \bullet \frac{\sqrt{7}}{\sqrt{7}} = \frac{2\sqrt{7}}{\sqrt{49}}$ ← Because I remember the perfect square roots from the chart on page 240, I see that $\sqrt{49}$ is a perfect square root…and therefore rational!

$\frac{2}{\sqrt{7}} = \frac{2\sqrt{7}}{\sqrt{49}} = \frac{2\sqrt{7}}{7}$ ← This may *not* look like a simpler expression than I started with, but *it does conform to the second rule*.

Follow along with this example.

$$\frac{6}{\sqrt{3}} = \frac{6}{\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{\sqrt{9}} = \frac{6\sqrt{3}}{3} = \frac{2\sqrt{3}}{1} = 2\sqrt{3}$$

In the above example, notice that we reduced the "real 6" and the "real 3," but not with the square root of 3. Do *not* mix a rational number with a *nonrational* number when you are reducing…they are *not* **like terms**!

It's time for you to practice.

Practice

Simplify *each* **radical expression**.

Remember: Never leave a square root in the denominator.

Example: $\frac{6}{\sqrt{5}} = \frac{6}{\sqrt{5}} \bullet \frac{\sqrt{5}}{\sqrt{5}} = \frac{6\sqrt{5}}{\sqrt{25}} = \frac{6\sqrt{5}}{5}$

Show all your steps.

1. $\frac{7}{\sqrt{2}} =$

2. $\frac{5}{\sqrt{6}} =$

3. $\frac{1}{\sqrt{3}} =$

4. $\frac{3}{\sqrt{5}} =$

5. $\frac{5}{\sqrt{18}} =$

6. $\frac{4}{\sqrt{3}} =$

7. $\frac{7}{\sqrt{10}} =$

8. $\frac{3}{\sqrt{7}} =$

9. $\frac{4}{\sqrt{11}} =$

10. $\frac{\sqrt{2}}{\sqrt{15}} =$

Practice

Simplify *each* radical expression.

Example: $\frac{10}{\sqrt{6}} = \frac{10}{\sqrt{6}} \bullet \frac{\sqrt{6}}{\sqrt{6}} = \frac{10\sqrt{6}}{\sqrt{36}} = \frac{10\sqrt{6}}{6} = \frac{5\sqrt{6}}{3}$

1. $\frac{9}{\sqrt{6}} =$

2. $\frac{-2}{\sqrt{8}} =$

3. $\frac{2}{\sqrt{7}} =$

4. $\frac{5}{\sqrt{5}} =$

5. $\sqrt{\frac{2}{3}} =$

6. $\frac{3}{\sqrt{18}} =$

7. $\sqrt{\frac{1}{3}} =$

8. $\frac{-5}{\sqrt{20}} =$

9. $\frac{3\sqrt{5}}{\sqrt{6}} =$

10. $\frac{7\sqrt{3}}{\sqrt{5}} =$

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. a number whose square root is a whole number

_____ 2. a number under the radical sign that is not a product with a perfect square factor

_____ 3. the symbol ($\sqrt{\ }$) used before a number to show that the number is a *radicand*

_____ 4. terms that have the same variables and the same corresponding exponents

_____ 5. a real number that cannot be expressed as a ratio of two integers

_____ 6. a number or expression that divides evenly into another number

_____ 7. a numerical expression containing a radical sign

_____ 8. a real number that can be expressed as a ratio of two integers

_____ 9. a positive real number that can be multiplied by itself to produce a given number

A. factor

B. irrational number

C. like terms

D. perfect square

E. radical expression

F. radical sign

G. rational number

H. simplest radical form

I. square root

Lesson Two Purpose

- Add, subtract, multiply and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Addition and Subtraction of Radical Expressions

We can add or subtract radical expressions only when those radical expressions match. For instance,

$$5\sqrt{2} + 6\sqrt{2} = 11\sqrt{2}.$$

Notice that we did not change the $\sqrt{2}$'s. We simply added the **coefficients** because they had matching radical parts.

Remember: *Coefficients* are any factor in a **term**. Usually, but not always, a coefficient is a number instead of a **variable** or a *radical*.

The same is true when we subtract radical expressions.

$$5\sqrt{7} - 3\sqrt{7} = 2\sqrt{7}$$

Sometimes, at first glance, it may appear that there are no matching numbers under the radical sign. But, if we **simplify** the expressions, we often find radical expressions that we can add or subtract.

Look at this example.

$$3\sqrt{8} + 5\sqrt{2} - 4\sqrt{32}$$

Hopefully, you notice that $\sqrt{8}$ and $\sqrt{32}$ each have perfect square factors and can be simplified. Follow the simplication process step by step and see what happens.

$3\sqrt{8} + 5\sqrt{2} - 4\sqrt{32} =$

$3\sqrt{4}\sqrt{2} + 5\sqrt{2} - 4\sqrt{16}\sqrt{2} =$ ← We *found* the perfect square factors of $\sqrt{8}$ and $\sqrt{32}$ and rewrote the problem.

$3 \bullet 2\sqrt{2} + 5\sqrt{2} - 4 \bullet 4\sqrt{2} =$ ← Next, we *simplified* the perfect square roots.

$6\sqrt{2} + 5\sqrt{2} - 16\sqrt{2} =$ ← We *multiplied* the new factors for each coefficient.

$-5\sqrt{2}$ ← Finally, we *add and subtract* matching radical expressions, in order, from *left to right*.

When Radical Expressions Don't Match or Are Not in Radical Form

What happens when radical expressions don't match, or there is a number that is not in radical form? Just follow the steps on the previous pages and leave your answer with the appropriate terms. Watch this!

$$\sqrt{75} + \sqrt{27} - \sqrt{16} + \sqrt{80} =$$
$$\sqrt{25}\sqrt{3} + \sqrt{9}\sqrt{3} - 4 + \sqrt{16}\sqrt{5} =$$
$$5\sqrt{3} + 3\sqrt{3} - 4 + 4\sqrt{5} =$$
$$8\sqrt{3} - 4 + 4\sqrt{5} =$$
$$8\sqrt{3} + 4\sqrt{5} - 4$$

Practice

Simplify *each of the following. Refer to pages 249-251 as needed.*

1. $4\sqrt{7} + 10\sqrt{7} =$

2. $-5\sqrt{2} + 7\sqrt{2} - 4\sqrt{2} =$

3. $3\sqrt{7} + 5 - \sqrt{7} =$

4. $2\sqrt{27} - 4\sqrt{12} =$

5. $\sqrt{2} + \sqrt{18} - \sqrt{16} =$

6. $\sqrt{3} + 5\sqrt{3} - \sqrt{27} =$

7. $\sqrt{50} + \sqrt{18} =$

8. $\sqrt{27} + \sqrt{12} - \sqrt{48} =$

Practice

Simplify *each of the following. Refer to pages 249-251 as needed.*

1. $-3\sqrt{5} + 4\sqrt{2} - \sqrt{5} + \sqrt{8} =$

2. $\sqrt{81} + \sqrt{24} - \sqrt{9} + \sqrt{54} =$

3. $\sqrt{50} - \sqrt{45} + \sqrt{32} - \sqrt{80} =$

4. $5\sqrt{7} + 2\sqrt{3} - 4\sqrt{7} - \sqrt{27} =$

5. $\sqrt{200} - \sqrt{8} + 3\sqrt{72} - 6 =$

6. $12 - 3\sqrt{5} + 2\sqrt{144} - \sqrt{20} =$

7. $\sqrt{18} + \sqrt{48} - \sqrt{32} - \sqrt{27} =$

Lesson Three Purpose

- Add, subtract, multiply and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Multiplication and Division of Radical Expressions

Radical expressions *don't* have to match when we multiply or divide them. The following examples show that we simply multiply or divide the **digits** under the radical signs and then simplify our results if possible.

Example 1

$$\sqrt{5} \times \sqrt{6} = \sqrt{30}$$

Example 2

$$\sqrt{8} \times \sqrt{3} = \sqrt{24} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$$

Example 3

$$\sqrt{18} \times \sqrt{2} = \sqrt{36} = 6$$

Example 4

$$\frac{\sqrt{12}}{\sqrt{3}} = \sqrt{4} = 2$$

Example 5

$$\frac{\sqrt{20}}{\sqrt{10}} = \sqrt{2}$$

Example 6

$$\frac{\sqrt{8}}{\sqrt{24}} = \frac{\sqrt{1}}{\sqrt{3}} \quad \text{(we must simplify this)} \quad \rightarrow \frac{\sqrt{1}}{\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{9}} = \frac{\sqrt{3}}{3}$$

After studying the examples above, try the following practice.

Practice

Simplify *each of the following. Refer to the* **examples** *on the previous page as needed.*

1. $\sqrt{5} \bullet \sqrt{10} =$

2. $\sqrt{2} \bullet \sqrt{50} =$

3. $\sqrt{75} \bullet \sqrt{3} =$

4. $\sqrt{6} \bullet \sqrt{10} =$

5. $\frac{\sqrt{30}}{\sqrt{2}} =$

6. $\frac{\sqrt{8}}{\sqrt{32}} =$

7. $\frac{\sqrt{6}}{\sqrt{10}} =$

8. $\frac{\sqrt{75}}{\sqrt{3}} =$

9. $\frac{\sqrt{72}}{\sqrt{18}} =$

10. $\frac{\sqrt{5}}{\sqrt{10}} =$

Working with a Coefficient for the Radical

What happens when there is a coefficient for the *radical*? Then it is important to remember to multiply or divide the radical numbers together separately from those coefficients. Don't forget to simplify each answer. Look at the following examples.

Example 1

$$3\sqrt{7} \bullet 5\sqrt{2} = 15\sqrt{14}$$

Example 2

$$6\sqrt{3} \bullet \sqrt{3} = 6\sqrt{9} = 6 \bullet 3 = 18$$

Remember: If there is *no* written coefficient, then it is understood to be a 1.

Example 3

$$\frac{2\sqrt{14}}{6\sqrt{7}} = \frac{1\sqrt{2}}{3} = \frac{\sqrt{2}}{3}$$

Example 4

$$\frac{12\sqrt{5}}{6\sqrt{10}} = \frac{2}{\sqrt{2}} \bullet \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{4}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

Example 5

$$\frac{\sqrt{6} - \sqrt{12}}{\sqrt{3}} = \frac{\sqrt{6}}{\sqrt{3}} - \frac{\sqrt{12}}{\sqrt{3}} = \sqrt{2} - \sqrt{4} = \sqrt{2} - 2$$

Now it's time to practice on the following page.

Practice

Simplify *each of the following. Refer to the* **examples** *on the previous page as needed.*

1. $5\sqrt{3} \bullet 6\sqrt{5} =$

2. $2\sqrt{5} \bullet 4\sqrt{2} =$

3. $8\sqrt{2} \bullet 5\sqrt{3} =$

4. $2\sqrt{7} \bullet \sqrt{7} =$

5. $\frac{3\sqrt{10}}{6\sqrt{5}} =$

6. $\frac{4\sqrt{6}}{2\sqrt{12}} =$

7. $\frac{9\sqrt{5}}{3\sqrt{10}} =$

8. $2\sqrt{7} \bullet 5\sqrt{7} =$

9. $5\sqrt{6} \bullet 4\sqrt{2} =$

Practice

Simplify *each of the following. Refer to the* **examples** *on page 259 as needed.*

1. $\frac{\sqrt{15}-\sqrt{20}}{\sqrt{5}} =$

2. $\frac{\sqrt{8}-\sqrt{12}}{\sqrt{2}} =$

3. $\frac{\sqrt{30}-\sqrt{50}}{\sqrt{10}} =$

4. $\frac{3\sqrt{18}}{\sqrt{3}} =$

5. $\frac{4\sqrt{6}}{6\sqrt{2}} =$

6. $\frac{10\sqrt{8}}{12\sqrt{12}} =$

7. $\frac{\sqrt{75}-\sqrt{50}}{\sqrt{25}} =$

Lesson Four Purpose

- Add, subtract, multiply and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Multiple Terms and Conjugates

Sometimes it is necessary to multiply or divide radical expressions with more than one *term*. To multiply radicals with multiple terms by a single term, we use the old reliable **distributive property**. Look at how the *distributive property* works for these examples.

Example 1

$$6(\sqrt{5} + \sqrt{3}) =$$
$$6\sqrt{5} + 6\sqrt{3}$$

Example 2

$$\sqrt{3}(2\sqrt{5} - 4\sqrt{3}) =$$
$$2\sqrt{15} - 4\sqrt{9} =$$
$$2\sqrt{15} - 4 \bullet 3 =$$
$$2\sqrt{15} - 12$$

Example 3

$$6\sqrt{3}(2\sqrt{2} + 5\sqrt{6}) =$$
$$12\sqrt{6} + 30\sqrt{18} =$$
$$12\sqrt{6} + 30\sqrt{9}\sqrt{2} =$$
$$12\sqrt{6} + 30 \bullet 3\sqrt{2} =$$
$$12\sqrt{6} + 90\sqrt{2}$$

Practice

Simplify *each of the following. Refer to the* **examples** *on the previous page as needed.*

1. $2(\sqrt{6} + \sqrt{5}) =$

2. $\sqrt{2}(\sqrt{6} + \sqrt{5}) =$

3. $3\sqrt{2}(5\sqrt{3} - 4\sqrt{2}) =$

4. $6(3\sqrt{8} - 5\sqrt{2}) =$

5. $\sqrt{6}(3\sqrt{8} - 5\sqrt{2}) =$

6. $-2(\sqrt{5} + 7) =$

7. $2\sqrt{3}(\sqrt{7} + \sqrt{10}) =$

8. $4(2\sqrt{3} - 5\sqrt{2}) =$

9. $4\sqrt{3}(2\sqrt{3} - 5\sqrt{2}) =$

10. $8\sqrt{6}(2\sqrt{6} + 5\sqrt{8}) =$

The FOIL Method

Another old reliable method we can use when multiplying two radical expressions with multiple terms is the **FOIL method** of multiplying the **f**irst, **o**utside, **i**nside, and **l**ast terms. We employ the same process in problems like these.

Example 1

$$(\sqrt{6} - 5)(\sqrt{3} + 4) =$$

$$\sqrt{6} \bullet \sqrt{3} + \sqrt{6} \bullet 4 - 5 \bullet \sqrt{3} - 5 \bullet 4 =$$

Multiply the **first** terms, the **outside** terms, the **inside** terms, and then the **last** terms.

$$\sqrt{18} + 4\sqrt{6} - 5\sqrt{3} - 20 =$$

Carefully write out the **products**.

$$3\sqrt{2} + 4\sqrt{6} - 5\sqrt{3} - 20$$

Simplify each term and combine like terms (if needed).

Example 2

$$(\sqrt{3} + \sqrt{2})(\sqrt{7} - \sqrt{11}) =$$

$$\sqrt{3}\sqrt{7} - \sqrt{3}\sqrt{11} + \sqrt{2}\sqrt{7} - \sqrt{2}\sqrt{11} =$$

$$\sqrt{21} - \sqrt{33} + \sqrt{14} - \sqrt{22}$$

Notice that no term has a perfect square as a factor. Therefore, there is no further simplifying to be done.

Time to try the following practice.

Practice

Simplify *each of the following. Refer to the* **examples** *on page 267 as needed.*

1. $(\sqrt{6} - 2)(\sqrt{5} + 7) =$

2. $(5 - \sqrt{3})(2 + \sqrt{7}) =$

3. $(4 + 5\sqrt{2})(2 - \sqrt{2}) =$

4. $(2\sqrt{5} - 3)(\sqrt{5} + 6) =$

5. $(4 - 3\sqrt{10})(2 - \sqrt{10}) =$

6. $(2\sqrt{7} - 3)(5\sqrt{7} + 1) =$

7. $(\sqrt{5} - 7)(3\sqrt{5} + 7) =$

8. $(\sqrt{10} - \sqrt{6})(\sqrt{7} - \sqrt{13}) =$

9. $(3\sqrt{6} + 2\sqrt{3})(\sqrt{5} - 2) =$

10. $(4\sqrt{3} - \sqrt{5})(3\sqrt{3} - \sqrt{5}) =$

11. $(3 + \sqrt{10})(3 - \sqrt{10}) =$

12. $(6\sqrt{5} + 4)(6\sqrt{5} - 4) =$

Two-Term Radical Expressions

At the beginning of this unit, we learned that there are two rules we must remember when simplifying a radical expression. Rule one required that we never leave a perfect square factor under a radical sign. Rule two insisted that we never leave a radical in the denominator. With that in mind, let's see what to do with two-term radical expressions.

In a problem like $\frac{2+\sqrt{7}}{5-\sqrt{6}}$, we see that we must rationalize the denominator (reformat it without using a square root). At first glance, it may seem to you that multiplying that denominator by itself makes the square roots disappear. But when we try that, we realize that new square roots appear as a result of the FOILing.

$$(5-\sqrt{6})(5-\sqrt{6}) =$$

$$25 - 5\sqrt{6} - 5\sqrt{6} + \sqrt{6}\sqrt{6} =$$

$$25 - 10\sqrt{6} + 6$$

So there must be a better way to rationalize this denominator. Try multiplying $(5-\sqrt{6})$ by its **conjugate** $(5+\sqrt{6})$. These numbers are *conjugates* because they match except for the signs between the terms. Notice that one has a "+" and the other has a "–".

$$(5-\sqrt{6})(5+\sqrt{6}) =$$

$$25 + 5\sqrt{6} - 5\sqrt{6} - \sqrt{6}\sqrt{6} =$$

$$25 - \sqrt{36} =$$

$$25 - 6 =$$

$$19$$

Remember, we only need to rationalize the denominator. It is acceptable to leave simplified square roots in the numerator. Now, let's take a look at the entire problem.

$$\frac{2+\sqrt{7}}{5-\sqrt{6}} \bullet \frac{5+\sqrt{6}}{5+\sqrt{6}} =$$

← reformat the fraction by multiplying it by 1

$$\frac{5+\sqrt{6}}{5+\sqrt{6}} = 1$$

$$\frac{(2)(5)+2\sqrt{6}+5\sqrt{7}+\sqrt{42}}{(5)(5)+5\sqrt{6}-5\sqrt{6}-\sqrt{6}\sqrt{6}} =$$

← FOIL the numerator and denominator

$$\frac{10+2\sqrt{6}+5\sqrt{7}+\sqrt{42}}{25-\sqrt{36}} =$$

← simplify

$$\frac{10+2\sqrt{6}+5\sqrt{7}+\sqrt{42}}{25-6} =$$

← simplify again

$$\frac{10+2\sqrt{6}+5\sqrt{7}+\sqrt{42}}{19}$$

← and again, if necessary

Follow along with this one!

$$\frac{3+\sqrt{2}}{4+\sqrt{8}} \bullet \frac{4-\sqrt{8}}{4-\sqrt{8}} =$$

← reformat the fraction by multiplying it by 1

$$\frac{4-\sqrt{8}}{4-\sqrt{8}} = 1$$

$$\frac{(3)(4)-3\sqrt{8}+4\sqrt{2}-\sqrt{16}}{(4)(4)-4\sqrt{8}+4\sqrt{8}-\sqrt{8}\sqrt{8}} =$$

← FOIL the numerator and denominator

$$\frac{12-3\sqrt{4}\sqrt{2}+4\sqrt{2}-\sqrt{16}}{16-\sqrt{64}} =$$

← simplify

$$\frac{12-3\bullet 2\sqrt{2}+4\sqrt{2}-4}{16-8} =$$

← simplify again

$$\frac{12-6\sqrt{2}+4\sqrt{2}-4}{8} =$$

← and again

$$\frac{8-2\sqrt{2}}{8} = \frac{2(4-\sqrt{2})}{8} =$$

← and again

$$\frac{4-\sqrt{2}}{4}$$

← and again, if necessary

With more practice, you will be able to mentally combine some of those simplifying steps and finish faster.

So let's practice on the following page.

Practice

Simplify *each of the following.*

1. $\frac{\sqrt{5}+2}{\sqrt{3}-1} =$

2. $\frac{\sqrt{6}+5}{3\sqrt{6}-2} =$

3. $\frac{5\sqrt{2}+7}{\sqrt{2}-3} =$

4. $\frac{\sqrt{7}-\sqrt{5}}{\sqrt{5}+\sqrt{7}} =$

5. $\frac{\sqrt{6}-\sqrt{3}}{\sqrt{6}+\sqrt{3}} =$

6. $\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}-5} =$

7. $\frac{\sqrt{5}+7}{\sqrt{5}-3} =$

8. $\frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}-3\sqrt{5}} =$

Practice

Simplify *each of the following.*

1. $\frac{4-\sqrt{7}}{3+\sqrt{7}} =$

2. $\frac{\sqrt{6}+5}{3\sqrt{6}-2} =$

3. $\frac{4\sqrt{2}-\sqrt{3}}{\sqrt{2}+3\sqrt{3}} =$

4. $\frac{6\sqrt{5}-2}{\sqrt{5}+\sqrt{2}} =$

5. $\frac{6\sqrt{2}+5}{1+\sqrt{5}} =$

6. $\frac{5+3\sqrt{2}}{1-\sqrt{2}} =$

7. $\frac{\sqrt{6}+2}{2\sqrt{6}+1} =$

8. $\frac{\sqrt{5}+2\sqrt{7}}{\sqrt{5}+\sqrt{7}} =$

Practice

Match each **symbol or expression** *with the appropriate* **description**.

_____	1. 7	A. coefficient in the expression $5\sqrt{x}$
_____	2. $\sqrt{}$	B. conjugate of $x + 4$
_____	3. $\frac{2}{\sqrt{7}} = \frac{2\sqrt{7}}{\sqrt{49}} = \frac{2\sqrt{7}}{7}$	C. perfect square of 11
_____	4. $x - 4$	D. radical expression
_____	5. 5	E. radical sign
_____	6. $3x\sqrt{6}$	F. rationalizing the denominator
_____	7. 121	G. square root of 49

Unit Review

Simplify *each of the following.*

1. $\sqrt{75} =$

2. $-\sqrt{40} =$

3. $5\sqrt{27} =$

4. $\frac{3}{\sqrt{36}} =$

5. $\frac{5}{\sqrt{8}} =$

6. $\frac{1}{\sqrt{7}} =$

7. $\sqrt{\frac{3}{8}} =$

8. $\frac{5\sqrt{6}}{\sqrt{5}} =$

9. $6\sqrt{3} - 8\sqrt{3} =$

10. $4\sqrt{8} - 5\sqrt{2} + 3\sqrt{32} =$

11. $\sqrt{75} - \sqrt{45} - \sqrt{80} =$

12. $2\sqrt{50} - 3\sqrt{45} + \sqrt{32} + \sqrt{80} =$

13. $\sqrt{5} + \sqrt{2} + \sqrt{8} + \sqrt{125} =$

14. $\sqrt{2} \times \sqrt{10} =$

15. $\frac{\sqrt{18}}{\sqrt{3}} =$

16. $\sqrt{6} \bullet \sqrt{2} =$

17. $\frac{2\sqrt{6}}{\sqrt{24}} =$

18. $\sqrt{\frac{6}{30}} =$

19. $\frac{\sqrt{60}}{3\sqrt{5}} =$

20. $8\sqrt{3} \bullet 2\sqrt{8} =$

21. $\frac{\sqrt{18}-\sqrt{12}}{\sqrt{6}} =$

22. $(3+5\sqrt{6})(3-5\sqrt{6}) =$

23. $\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}} =$

24. $\frac{5+2\sqrt{3}}{2+\sqrt{5}} =$

25. $\frac{\sqrt{6}-1}{2\sqrt{6}+2} =$

Unit 5: What about That Pythagorean Theorem?

This unit explores the Pythagorean theorem, its special cases, and real-world applications.

Unit Focus

Number Sense, Concepts, and Operations

- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)

Geometry and Spatial Sense

- Represent and apply geometric properties and relationships to solve real-world and mathematical problems including ratio and proportion. (MA.C.3.4.1)

Vocabulary

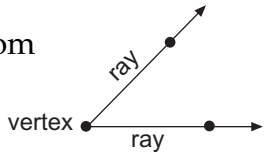
Use the vocabulary words and definitions below as a reference for this unit.

30°-60°-90° triangle ..... a triangle with angles that measure 30°, 60°, and 90°

45°-45°-90° triangle ..... a triangle with angles that measure 45°, 45°, and 90°

altitude ..... the perpendicular distance from a vertex in a polygon to its opposite side

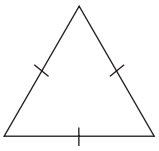
angle (∠) ..... two rays extending from a common endpoint called the vertex; measures of angles are described in degrees (°)



degree (°) ..... common unit used in measuring angles

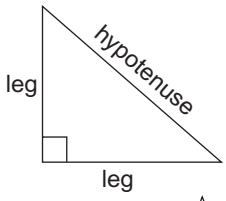
diagonal (of a polygon) ..... a line segment that joins two vertices of a polygon but is not a side of the polygon

equilateral triangle ..... a triangle with three congruent sides

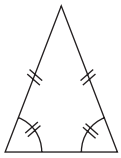


exponent (exponential form) ..... the number of times the base occurs as a factor
Example: 2^3 is the exponential form of 2 x 2 x 2. The numeral two (2) is called the *base*, and the numeral three (3) is called the *exponent*.

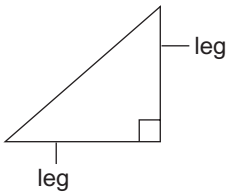
hypotenuse ..... the longest side of a right triangle; the side opposite the right angle



isosceles triangle ..... a triangle with two congruent sides and two congruent angles



leg ..... in a right triangle, one of the two sides that form the right angle



length (*l*) ..... a one-dimensional measure that is the measurable property of line segments

line segment (—) ..... a portion of a line that consists of two defined endpoints and all the points in between
Example: The line segment *AB* is between point *A* and point *B* and includes point *A* and point *B*.

opposite sides ..... sides that are directly across from each other

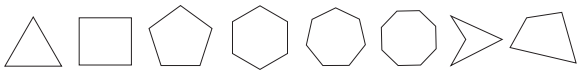
order of operations ..... the order of performing computations in parentheses first, then exponents or powers, followed by multiplication and/or division (as read from left to right), then addition and/or subtraction (as read from left to right); also called *algebraic order of operations*

Example: $5 + (12 - 2) \div 2 - 3 \times 2 =$
$5 + 10 \div 2 - 3 \times 2 =$
$5 + 5 - 6 =$
$10 - 6 =$
4

perimeter (*P*) ..... the distance around a polygon

polygon ..... a closed-plane figure, having at least three sides that are line segments and are connected at their endpoints

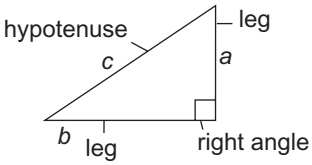
Example: triangle (3 sides), quadrilateral (4 sides), pentagon (5 sides), hexagon (6 sides), heptagon (7 sides), octagon (8 sides); concave, convex



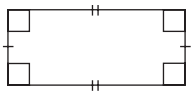
power (of a number) ..... an exponent; the number that tells how many times a number is used as a factor

Example: In 2^3, 3 is the power.

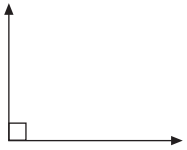
Pythagorean theorem ..... the square of the hypotenuse (c) of a right triangle is equal to the sum of the squares of the legs (a and b), as shown in the equation $c^2 = a^2 + b^2$



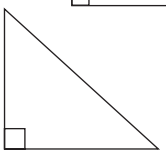
rectangle ..... a parallelogram with four right angles



right angle ..... an angle whose measure is exactly 90°

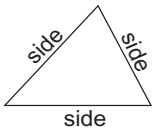


right triangle ..... a triangle with one right angle

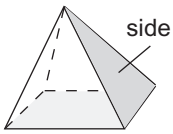


root ..... an equal factor of a number
Example:
In $\sqrt{144} = 12$, 12 is the square root.
In $\sqrt[3]{125} = 5$, 5 is the cube root.

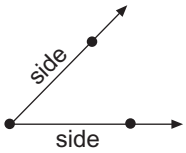
side ..... the edge of a polygon, the face of a polyhedron, or one of the rays that make up an angle
Example: A triangle has three sides.



edge of a polygon

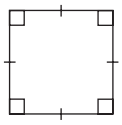


face of a polyhedron



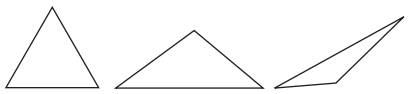
ray of an angle

square ..... a rectangle with four sides the same length



square root ..... a positive real number that can be multiplied by itself to produce a given number
Example: The square root of 144 is 12 or $\sqrt{144} = 12$.

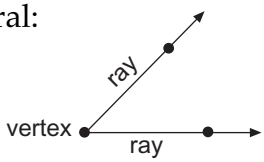
triangle ..... a polygon with three sides; the sum of the measures of the angles is 180°



value (of a variable) ..... any of the numbers represented by the variable

variable ..... any symbol, usually a letter, which could represent a number

vertex ..... the point common to the two rays that form an angle; the point common to any two sides of a polygon; the point common to three or more edges of a polyhedron; (plural: *vertices*); vertices are named clockwise or counterclockwise



Unit 5: What about That Pythagorean Theorem?

Introduction

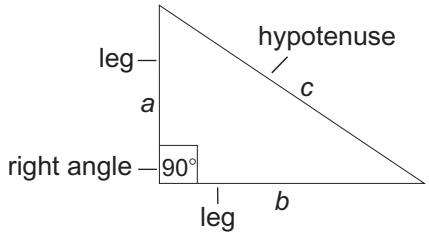
Students should be able to recognize and employ the Pythagorean theorem. By knowing a few facts beyond the theorem itself, students can solve many problems mentally.

Lesson One Purpose

- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Represent and apply geometric properties and relationships to solve real-world and mathematical problems including ratio and proportion. (MA.C.3.4.1)

Pythagorean Theorem

A **right triangle** is a **triangle** with a 90 **degree (°)** or **right angle** in it. This *right angle* is often indicated with a tiny box in the corner of the **angle (∠)**. The two **sides** of the *triangle* that meet at the right angle are called the **legs** of the triangle. The remaining *side* is called the **hypotenuse**. Notice in the figure below that the *hypotenuse* is across the triangle from the right angle. It is also the longest side of the triangle.



A right triangle—a triangle with a 90° or right angle in it.

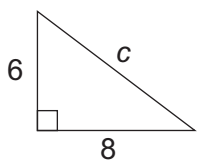
A special relationship exists among the three sides of a *right triangle*. That relationship is called the **Pythagorean theorem**.

> **Pythagorean Theorem**
>
> *Pythagorean theorem*—In a right triangle, $a^2 + b^2 = c^2$, if a and b are the *legs* and c is the hypotenuse.

Look at the following examples. Watch out for the **order of operations**! For a quick review of the rules for the *order of operations*, see below.

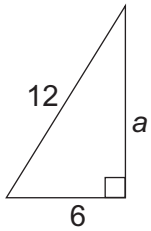
Example 1

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 6^2 + 8^2 &= c^2 \\ 36 + 64 &= c^2 \\ 100 &= c^2 \\ 10 &= c \end{aligned}$$



Example 2

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 6^2 &= 12^2 \\ a^2 + 36 &= 144 \\ a^2 + 36 - 36 &= 144 - 36 \\ a^2 &= 108 \\ a &= \sqrt{108} \\ a &= \sqrt{36}\sqrt{3} \\ a &= 6\sqrt{3} \end{aligned}$$

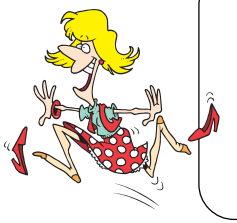


Rules for Order of Operations

Always start on the *left* and move *to* the *right*.

1.	Do operations inside *parentheses* first.	( ), [ ], **or** $\frac{x}{y}$
2.	Then do all *powers* (exponents) **or** *roots*.	x^2 **or** $\sqrt{x}$
3.	Next do *multiplication* **or** *division*— as they occur from left to right.	x **or** ÷
4.	Finally, do *addition* **or** *subtraction*— as they occur from left to right.	+ **or** —

Some people remember these rules by using this mnemonic device to help their memory.



Please **P**ardon **M**y **D**ear **A**unt **S**ally*

Please ..... **P**arentheses
Pardon ..... **P**owers
My **D**ear ..... **M**ultiplication or **D**ivision
Aunt **S**ally ..... **A**ddition or **S**ubtraction

*Also known as **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally—**P**arentheses, **E**xponents, **M**ultiplication or **D**ivision, **A**ddition or **S**ubtraction.

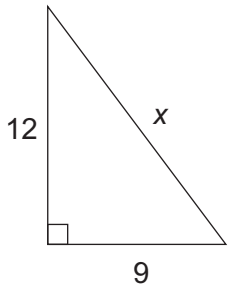
Try the problems in the following practice.

Practice

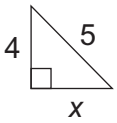
Use the **Pythagorean theorem** *below to find the* **value of the variable** *for each* **right triangle**. *Write answers in simplest* **square root** *form.*

Remember: A **variable** is any symbol, usually a letter, which could represent a number.

Pythagorean theorem
$a^2 + b^2 = c^2$

1. 

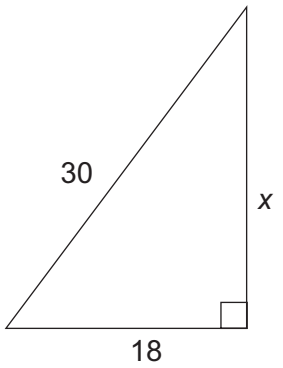
Answer: _____

2. 

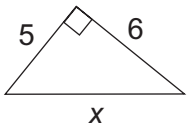
Answer: _____

3. 

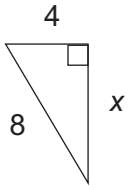
Answer: _____

4. 

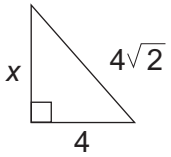
Answer: _____

5. 

Answer: _____

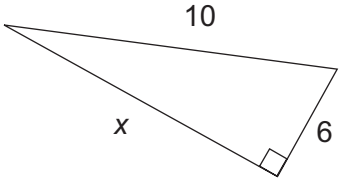
6. 

Answer: _____

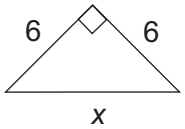
7. 

Hint: $(4\sqrt{2})^2 =$
$16 \bullet 2 =$
32

Answer: _____

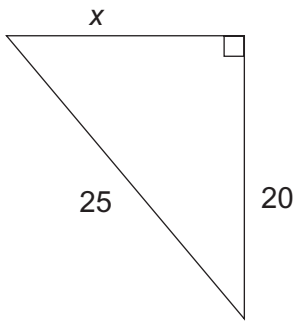
8. 

Answer: _____

9. 

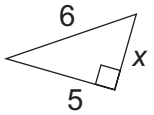
Answer: _____

10.



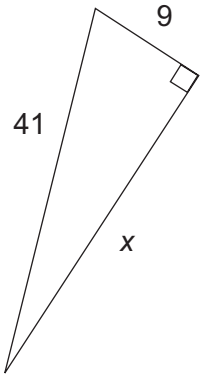
Answer: _____

11.



Answer: _____

12.



Answer: _____

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. the edge of a polygon

_____ 2. an angle whose measure is exactly 90°

_____ 3. a triangle with one right angle

_____ 4. the order of performing computations in parentheses first, then exponents or powers, followed by multiplication and/or division (as read from left to right), then addition and/or subtraction (as read from left to right)

_____ 5. in a right triangle, one of the two sides that form the right angle

_____ 6. the longest side of a right triangle; the side opposite the right angle

_____ 7. the square of the hypotenuse (c) of a right triangle is equal to the sum of the squares of the legs (a and b), as shown in the equation $c^2 = a^2 + b^2$

_____ 8. a polygon with three sides; the sum of the measures of the angles is 180°

A. hypotenuse

B. leg

C. order of operations

D. Pythagorean theorem

E. right angle

F. right triangle

G. side

H. triangle

Using the Pythagorean Theorem to Solve Problems

Many word problems that deal with geometric shapes can be solved using the Pythagorean theorem. It is helpful to draw the figure mentioned in the problem. Then determine if you can make a right triangle from the situation and use the Pythagorean theorem to solve for the missing **length (*l*)**. Always reread the problem to see if you need to go a step or two further to find the answer to the problem.

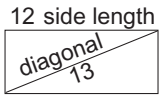
Let's look at an example.

Example

The **diagonal** of a **rectangle** is 13 inches. The *length* of one side of the *rectangle* is 12 inches. Find the **perimeter (*P*)** of the rectangle.

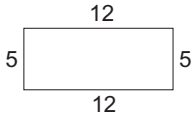
Step 1: Draw a rectangle. Remember that to find *perimeter* of the rectangle, or distance around it, we need to know the lengths of all sides.

Step 2: Put in the *diagonal*, or the **line segment (—)** that joins two **vertices** of a **polygon**. (**Note:** The diagonal is *not* a side of the *polygon*.) You now need to find the length of the side of the rectangle.



Step 3: Use the Pythagorean theorem to solve for the missing side of the triangle.

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
a^2 + 12^2 &= 13^2 \\
a^2 + 144 &= 169 \\
a^2 + 144 - 144 &= 169 - 144 \\
a^2 &= 25 \\
a &= \sqrt{25} \\
a &= 5
\end{aligned}
$$



Step 4: Reread the problem to see that we are looking for the perimeter. Because the lengths of the **opposite sides** of a rectangle are equal in length, we know all the information necessary to find the perimeter.

$$5 + 12 + 5 + 12 = \text{perimeter}$$
$$34 = \text{perimeter}$$

Answer: 34 inches = perimeter

Practice

Use the **Pythagorean theorem** *below and* **draw diagrams** *to help you solve the following. Leave answers in simplest* **square root** *form. Refer to page 299-300 as needed.*

Pythagorean theorem
$a^2 + b^2 = c^2$

Perimeter is the distance around a polygon.

1. The side of a **square** is 3 feet long. Find the length of the diagonal.

   Answer: _____ feet

2. The perimeter of a *square* is 16 yards. Find the length of the diagonal.

   Answer: _____ yards

3. The diagonal of a rectangle is 15 meters. The length of a side is 9 meters. Find the perimeter of the rectangle.

   Answer: _____ meters

4. Georgio drives 10 miles east, then 16 miles north, then 4 miles north again. Find the distance from his starting point to his ending point.
   (**Hint:** Distance is along a straight line. Redraw his route to make one right triangle, then solve.)

   Answer: _____ miles

5. Felicia walks 3 blocks west, 4 blocks south, 3 more blocks west, then 2 blocks south again. How far is Felicia from her starting point?

   Answer: _____ blocks

6. Find the perimeter of a right triangle if one leg is 4 inches and the other is 3 inches.

   Answer: _____ inches

7. The diagonal of a rectangle is 10 inches and the length of a side is 6 inches. Find the perimeter of the rectangle.

   Answer: _____ inches

8. A man drives 20 miles north, then 30 miles east, then 20 more miles north. How far is he from his starting point?

   Answer: _____ miles

Practice

Use the list below to write the correct term for each definition on the line provided.

diagonal (of a polygon)	**perimeter (*P*)**
length (*l*)	**rectangle**
opposite sides	**square**

_____ 1. sides that are directly across from each other

_____ 2. the distance around a polygon

_____ 3. a line segment that joins two vertices of a polygon but is not a side of the polygon

_____ 4. a parallelogram with four right angles

_____ 5. a one-dimensional measure that is the measurable property of line segments

_____ 6. a rectangle with four sides the same length

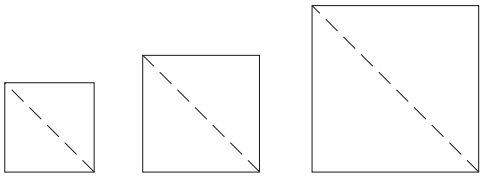
Lesson Two Purpose

- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Represent and apply geometric properties and relationships to solve real-world and mathematical problems including ratio and proportion. (MA.C.3.4.1)

Special Right Triangles

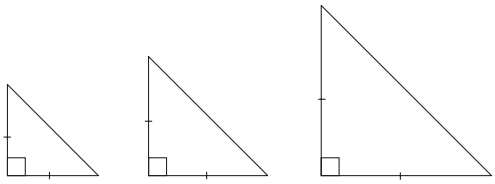
When we divide a square in half with a diagonal or an **equilateral triangle** in half with an **altitude**, special right triangles are formed. Follow the directions below to see what happens.

- For each square, choose a length for the sides and then use the Pythagorean theorem to find the length of the diagonal for each square.



- For each **isosceles right triangle** below, choose a length for the legs and then find the length of the hypotenuse.

Notice that each triangle is half a square.



After carefully looking at the results above, do you see a *relationship* between the lengths of the legs and the hypotenuse? What do you think the rule might be?

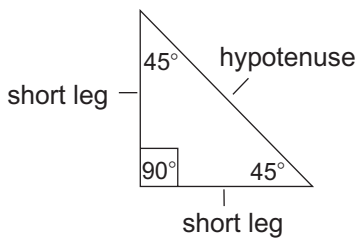
Read the top of the next page to see if you are correct.

In each case, the hypotenuse should have been $\sqrt{2}$ times the length of the leg. We call these right *isosceles triangles* **45°-45°-90° triangles** because of the measures of the angles. Because there is a special relationship between the lengths of the legs and the hypotenuse in a *45°-45°-90° triangle*, you can find the missing side without using the Pythagorean theorem.

The Shortcut

In a 45°-45°-90° triangle, the legs are always equal in length and the hypotenuse is the length of one leg times $\sqrt{2}$. If you know the length of the hypotenuse, you can find the length of a leg by dividing by $\sqrt{2}$ and simplifying.

45°-45°-90° Triangle

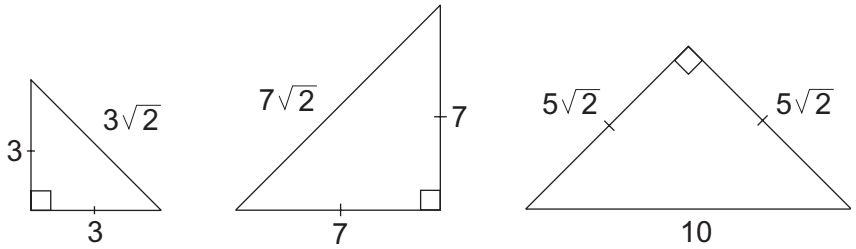


45°-45°-90° triangle

- the short legs are equal in length
- the hypotenuse is the length of one leg times $\sqrt{2}$

  or
- the length of the hypotenuse can be found by dividing one leg by $\sqrt{2}$ and simplifying

Look at these examples.



Use the shortcut described above to solve the lengths in the following practice.

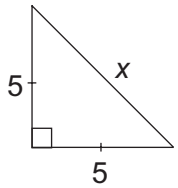
Practice

Use what you know about **45°-45°-90° triangles** *to solve the following. Refer to pages 305 and 306 as needed.*

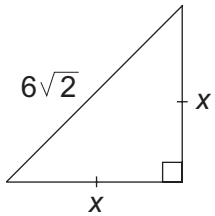
45°-45°-90° triangle

- the short legs are equal in length
- the hypotenuse is the length of one leg times $\sqrt{2}$

  or
- the length of the hypotenuse can be found by dividing one leg by $\sqrt{2}$ and simplifying

1. 

Answer: _____

2. 

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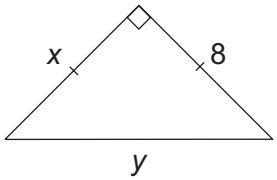
3. 

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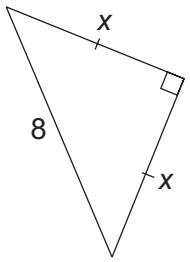
4. 

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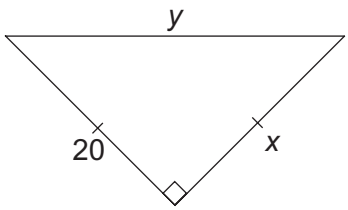
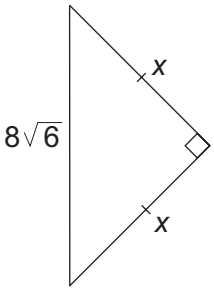
5. 

Answer: _____

6. 

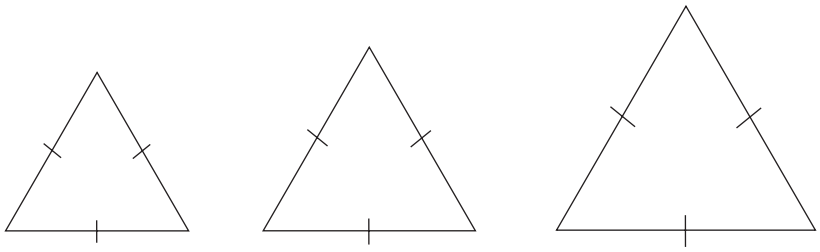
Answer: _____

More Special Right Triangles

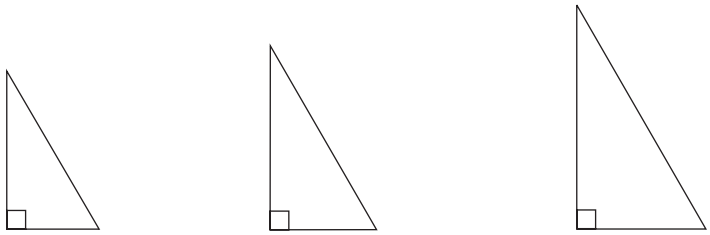
Follow the directions below.

- For each equilateral triangle below, choose a length for the sides (I suggest even numbers).
- Find the length of each altitude using the Pythagorean theorem.

The *altitude* divides the triangle exactly in half into two right triangles.



- For each half of an equilateral triangle below, choose a length for each hypotenuse (again, even numbers work better). Using the pattern from the triangles above, find the other lengths.



After carefully looking at the results above, do you see a relationship between the lengths of the legs and the hypotenuse? What do you think the rule might be?

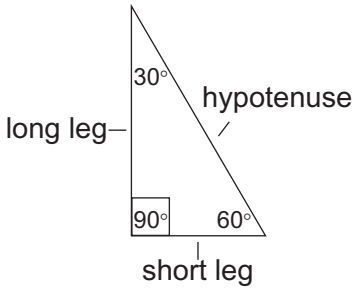
Read the top of the next page to see if you are correct.

In each case, the hypotenuse should have been twice as long as the shorter leg and the longer leg should have been $\sqrt{3}$ times the shorter leg. We call this type of triangle a **30°-60°-90° triangle**, because that's how big the angles are. Because there is a special relationship between the lengths of the legs and the hypotenuse in a *30°-60°-90° triangle*, you can find the missing sides without using the Pythagorean theorem. When you know the triangle is a 30°-60°-90° triangle, you only need to know the length of one side and you can find the others by following the pattern.

The Shortcut

In a 30°-60°-90° triangle, the *short* leg is always opposite the 30° angle and is half the length of the hypotenuse. The *long* leg is always opposite the 60° angle, and is $\sqrt{3}$ times the length of the short leg. And in reverse, the length of the short leg can be found by dividing the length of the long leg by $\sqrt{3}$.

30°-60°-90° Triangle



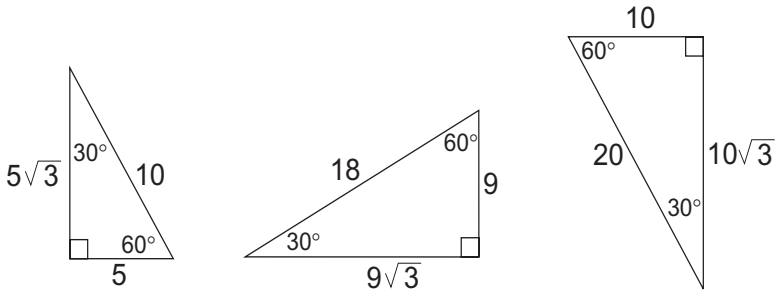
30°-60°-90° triangle

- the short leg is always opposite the 30° angle and is $\frac{1}{2}$ the length of the hypotenuse
- the long leg is always opposite the 60° angle and is $\sqrt{3}$ times the length of the short leg

  or
- the length of the short leg can be found by dividing the length of the long leg by $\sqrt{3}$

When working from the hypotenuse to the long leg or from the long leg to the hypotenuse, use the shortcut to get to the short leg and work from there.

Look at these examples.



Practice

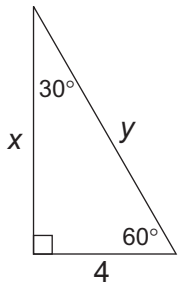
Use what you know about **30°-60°-90° triangles** *to solve the following. Refer to pages 309-311 as needed.*

30°-60°-90° triangle

- the short leg is always opposite the 30° angle and is $\frac{1}{2}$ the length of the hypotenuse
- the long leg is always opposite the 60° angle and is $\sqrt{3}$ times the length of the short leg

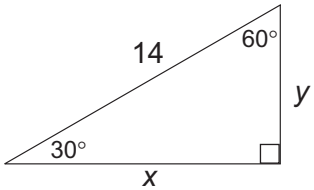
  or
- the length of the short leg can be found by dividing the length of the long leg by $\sqrt{3}$

1.



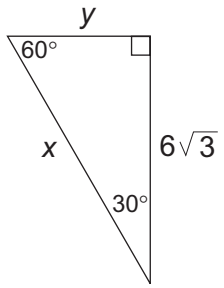
Answer: _____

2.



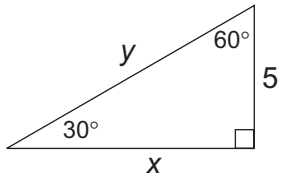
Answer: _____

3.



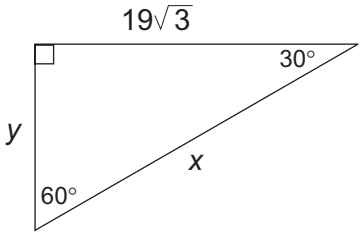
Answer: _____

4.

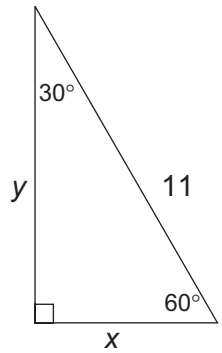


Answer: _____

5.



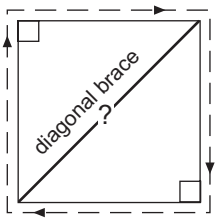
Answer: _____

6. 

Answer: _____

7. Pierre wants to put a *diagonal* brace across the back of a *square* picture frame. If the *perimeter* of the frame is 24 inches, how long must the brace be?

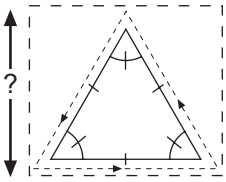
Answer: _____ inches



square picture frame with a perimeter of 24 inches

8. Elizabeth wants to make an *equilateral triangle* from a piece of poster board. She wants the *perimeter* of the triangle to be 36 inches. How tall must the poster board be for her to cut her triangle from it?

Answer: _____ inches



equilateral triangle with a perimeter of 36 inches

Practice

Use the list below to complete the following statements.

30°-60°-90° triangle	**isosceles**	**rectangle**
45°-45°-90° triangle	**leg**	**right angle**
diagonal	**perimeter**	**right triangle**
equilateral	**Pythagorean theorem**	**square**
hypotenuse		

1. A _____ is a polygon with four sides and four right angles.

2. A triangle with two sides the same length is an _____ triangle.

3. An angle whose measure is 90° is called a _____ .

4. A _____ is a line segment that connects two vertices of a polygon but is not a side of the polygon.

5. A right triangle whose hypotenuse is twice the short leg and whose long leg is $\sqrt{3}$ times the short leg is called a _____ triangle.

6. Any triangle that contains a right angle is called a _____ .

7. Another name for an isosceles right triangle is a _____ triangle.

8. A _____ is one of the two sides that form the right triangle.

9. The distance around a polygon is called the _____ .

10. A _____ is a rectangle whose sides all have the same length.

11. In a right triangle, the _____ is located across from the right angle.

12. A triangle with three congruent sides is an _____ triangle.

13. The _____ describes a special relationship that exists among three sides of a right triangle and is used to solve for a missing length in a right triangle.

Unit Review

Use the following to find the **missing lengths**. *Leave answers in simplest* **square root** *form.*

Pythagorean theorem
$a^2 + b^2 = c^2$

45°-45°-90° triangle

- the short legs are equal in length
- the hypotenuse is the length of one leg times $\sqrt{2}$

or

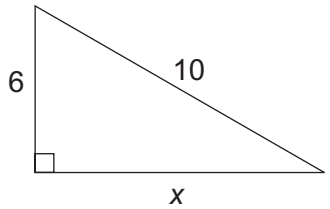
- the length of the hypotenuse can be found by dividing one leg by $\sqrt{2}$ and simplifying

30°-60°-90° triangle

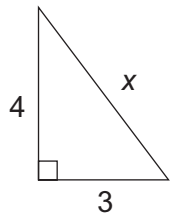
- the short leg is always opposite the 30° angle and is $\frac{1}{2}$ the length of the hypotenuse
- the long leg is always opposite the 60° angle and is $\sqrt{3}$ times the length of the short leg

or

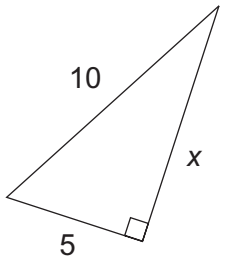
- the length of the short leg can be found by dividing the length of the long leg by $\sqrt{3}$

1. 

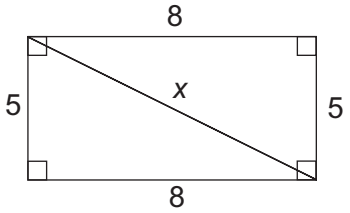
Answer: _____

2. 

Answer: _____

3. 

Answer: _____

4. 

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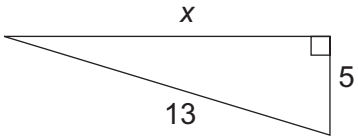
5. 

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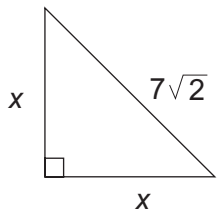
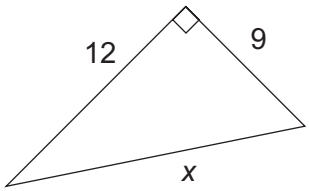


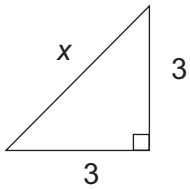
Answer: _____

7.



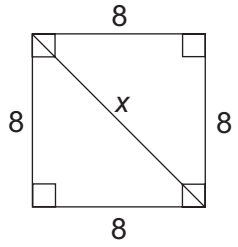
Answer: _____

8.



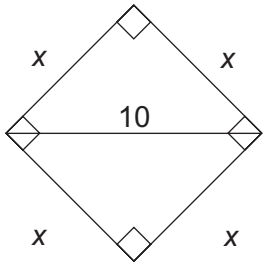
Answer: _____

9.



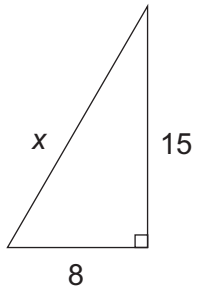
Answer: _____

10.



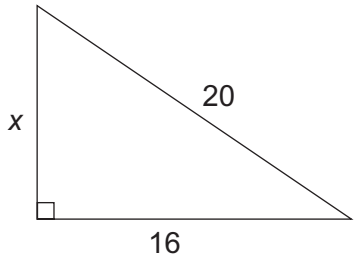
Answer: _____

11.



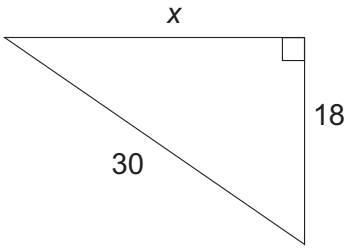
Answer: _____

12.



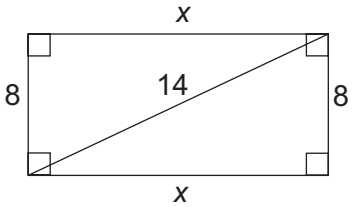
Answer: _____

13.



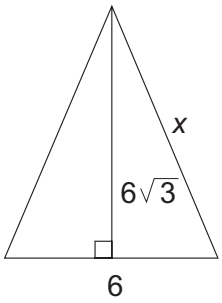
Answer: _____

14.



Answer: _____

15.



Answer: _____

Answer the following. Show all your work or *explain in words. Leave answers in simplest* **square root** *form.*

Pythagorean theorem
$a^2 + b^2 = c^2$

Perimeter is the distance around a polygon.

16. The perimeter of a square is 40 inches. How long is the diagonal of the square?

Answer: _____ inches

17. The diagonal of a rectangle is 17 feet. If the width of the rectangle is 8 feet, find the perimeter of the rectangle.

Answer: _____ inches

18. Kim left home and walked 4 blocks north. She turned east and walked 8 blocks, then turned back north and walked 2 more blocks. If she had walked in a straight line from home to her destination, how far would she have walked?

Answer: _____ blocks

19. Roberto wants to put a diagonal brace across a square gate. If the sides of the gate are 3 feet each, how long will the brace be?

Answer: _____ feet

20. In a 30°-60°-90° triangle, if the hypotenuse is 20, find the length of the long leg.

Answer: _____

Unit 6: Is There a Point to This?

This unit focuses on the concepts related to coordinate geometry.

Unit Focus

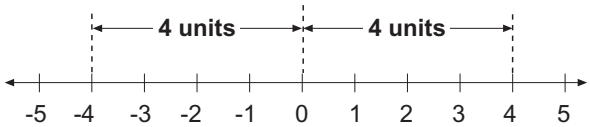
Geometry and Spatial Sense

- Understand geometric concepts such as perpendicularity, parallelism, congruency, reflections, symmetry, and transformations including flips, slides, turns, enlargements, and rotations. (MA.C.2.4.1)
- Using a rectangular coordinate system (graph), apply and algebraically verify properties of two-dimensional figures, including distance, midpoint, slope, parallelism, and perpendicularity. (MA.C.3.4.2)

Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

absolute value ..... a number's distance from zero (0) on a number line; distance expressed as a positive value
Example: The absolute value of both 4, written $|4|$, and negative 4, written $|-4|$, equals 4.



coordinate ..... the number paired with a point on the number line

coordinate grid or plane ..... a two-dimensional network of horizontal and vertical lines that are parallel and evenly-spaced; especially designed for locating points, displaying data, or drawing maps

coordinate plane ..... the plane containing the x- and y-axes

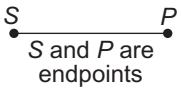
degree (°) ..... common unit used in measuring angles

denominator ..... the bottom number of a fraction, indicating the number of equal parts a whole was divided into
Example: In the fraction $\frac{2}{3}$ the denominator is 3, meaning the whole was divided into 3 equal parts.

diagonal (of a polygon) ..... a line segment that joins two vertices of a polygon but is not a side of the polygon

distance ..... the length of a segment connecting two points

endpoint ..... either of two points marking the end of a line segment



equation ..... a mathematical sentence in which two expressions are connected by an equality symbol
Example: $2x = 10$

expression ..... a collection of numbers, symbols, and/or operation signs that stands for a number
Example: $4r^2$; $3x + 2y$; $\sqrt{25}$
Expressions do *not* contain equality (=) or inequality (<, >, ≤, ≥, or ≠) symbols.

factor ..... a number or expression that divides evenly into another number; one of the numbers multiplied to get a product
Example: 1, 2, 4, 5, 10, and 20 are factors of 20 and $(x + 1)$ is one of the factors of $(x^2 - 1)$.

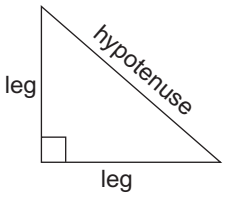
formula ..... a way of expressing a relationship using variables or symbols that represent numbers

graph ..... a drawing used to represent data
Example: bar graphs, double bar graphs, circle graphs, and line graphs

graph of a point ..... the point assigned to an ordered pair on a coordinate plane

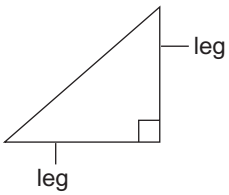
horizontal ..... parallel to or in the same plane of the horizon

hypotenuse ..... the longest side of a right triangle; the side opposite the right angle



intersect ..... to meet or cross at one point

leg ..... in a right triangle, one of the two sides that form the right angle



length (*l*) ..... a one-dimensional measure that is the measurable property of line segments

line (↔) ..... a collection of an infinite number of points in a straight pathway with unlimited length and having no width

linear equation ..... an algebraic equation in which the variable quantity or quantities are raised to the zero or first power and the graph is a straight line
Example: $20 = 2(w + 4) + 2w$; $y = 3x + 4$

line segment (—) ..... a portion of a line that consists of two defined endpoints and all the points in between
Example: The line segment *AB* is between point *A* and point *B* and includes point *A* and point *B*.

midpoint (of a line segment) ..... the point on a line segment equidistant from the endpoints

negative numbers ..... numbers less than zero

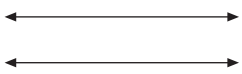
number line ..... a line on which ordered numbers can be written or visualized

numerator ..... the top number of a fraction, indicating the number of equal parts being considered
Example: In the fraction $\frac{2}{3}$, the numerator is 2.

ordered pair ..... the location of a single point on a rectangular coordinate system where the first and second values represent the position relative to the x-axis and y-axis, respectively
Example: (x, y) or (3, -4)

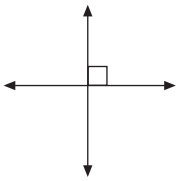
parallel (∥) ..... being an equal distance at every point so as to never intersect

parallel lines ..... two lines in the same plane that are a constant distance apart; lines with equal slopes



perpendicular (⊥) ..... two lines, two line segments, or two planes that intersect to form a right angle

perpendicular lines ..... two lines that intersect to form right angles

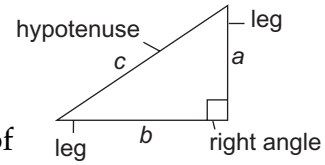


point ..... a specific location in space that has no discernable length or width

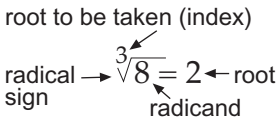
positive numbers ..... numbers greater than zero

product ..... the result of multiplying numbers together
Example: In 6 x 8 = 48,
48 is the product.

Pythagorean theorem ..... the square of the hypotenuse (c) of a right triangle is equal to the sum of the square of the legs (a and b), as shown in the equation $c^2 = a^2 + b^2$



radical ..... an expression that has a root (square root, cube root, etc.)
Example: $\sqrt{25}$ is a radical
Any root can be specified by an index number, b, in the form $\sqrt[b]{a}$ (e.g., $\sqrt[3]{8}$).
A radical without an index number is understood to be a square root.



radical

radical expression ..... a numerical expression containing a radical sign
Examples: $\sqrt{25}$ $\quad 2\sqrt{25}$

radical sign ($\sqrt{\ }$) ..... the symbol ($\sqrt{\ }$) used before a number to show that the number is a *radicand*

radicand ..... the number that appears within a radical sign

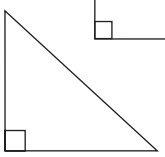
Example: In $\sqrt{25}$,
25 is the radicand.

reciprocals ..... two numbers whose product is 1; also called *multiplicative inverses*

Example: Since $\frac{3}{4} \times \frac{4}{3} = 1$, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.

right angle ..... an angle whose measure is exactly 90°

right triangle ..... a triangle with one right angle



rise ..... the vertical change on a graph between two points

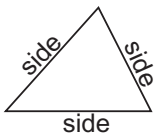
root ..... an equal factor of a number

Example:
In $\sqrt{144} = 12$, 12 is the square root.
In $\sqrt[3]{125} = 5$, 5 is the cube root.

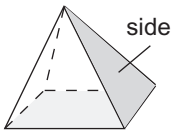
run ..... the horizontal change on a graph between two points

side ..... the edge of a polygon, the face of a polyhedron, or one of the rays that make up an angle

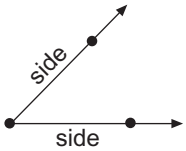
Example: A triangle has three sides.



edge of a polygon



face of a polyhedron



ray of an angle

simplest radical form ..... a number under the radical sign that is not a product with a perfect square factor
Example: $\sqrt{27} = \sqrt{9 \bullet 3} = \sqrt{9} \bullet \sqrt{3} = 3\sqrt{3}$

simplify a fraction ..... write fraction in lowest terms or simplest form

slope ..... the ratio of change in the vertical axis (y-axis) to each unit change in the horizontal axis (x-axis) in the form $\frac{\text{rise}}{\text{run}}$ or $\frac{\Delta y}{\Delta x}$; the constant, m, in the linear equation for the slope-intercept form $y = mx + b$

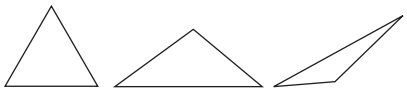
slope-intercept form ..... a form of a linear equation, $y = mx + b$, where m is the slope of the line and b is the y-intercept

square (of a number) ..... the result when a number is multiplied by itself or used as a factor twice
Example: 25 is the square of 5.

square root ..... a positive real number that can be multiplied by itself to produce a given number
Example: The square root of 144 is 12 or $\sqrt{144} = 12$.

sum ..... the result of adding numbers together
Example: In $6 + 8 = 14$,
14 is the sum.

triangle ..... a polygon with three sides



value (of a variable) ..... any of the numbers represented by the variable

variable ..... any symbol, usually a letter, which could represent a number

vertical ..... at right angles to the horizon; straight up and down

***x*-axis** ..... the horizontal number line on a rectangular coordinate system

***x*-coordinate** ..... the first number of an ordered pair

***y*-axis** ..... the vertical number line on a rectangular coordinate system

***y*-coordinate** ..... the second number of an ordered pair

Unit 6: Is There a Point to This?

Introduction

We will explore the relationships that exist between points, segments, and lines on a coordinate plane. Utilizing the formulas for finding distance, midpoint, and slope, we can identify the ways in which points and lines are related to each other.

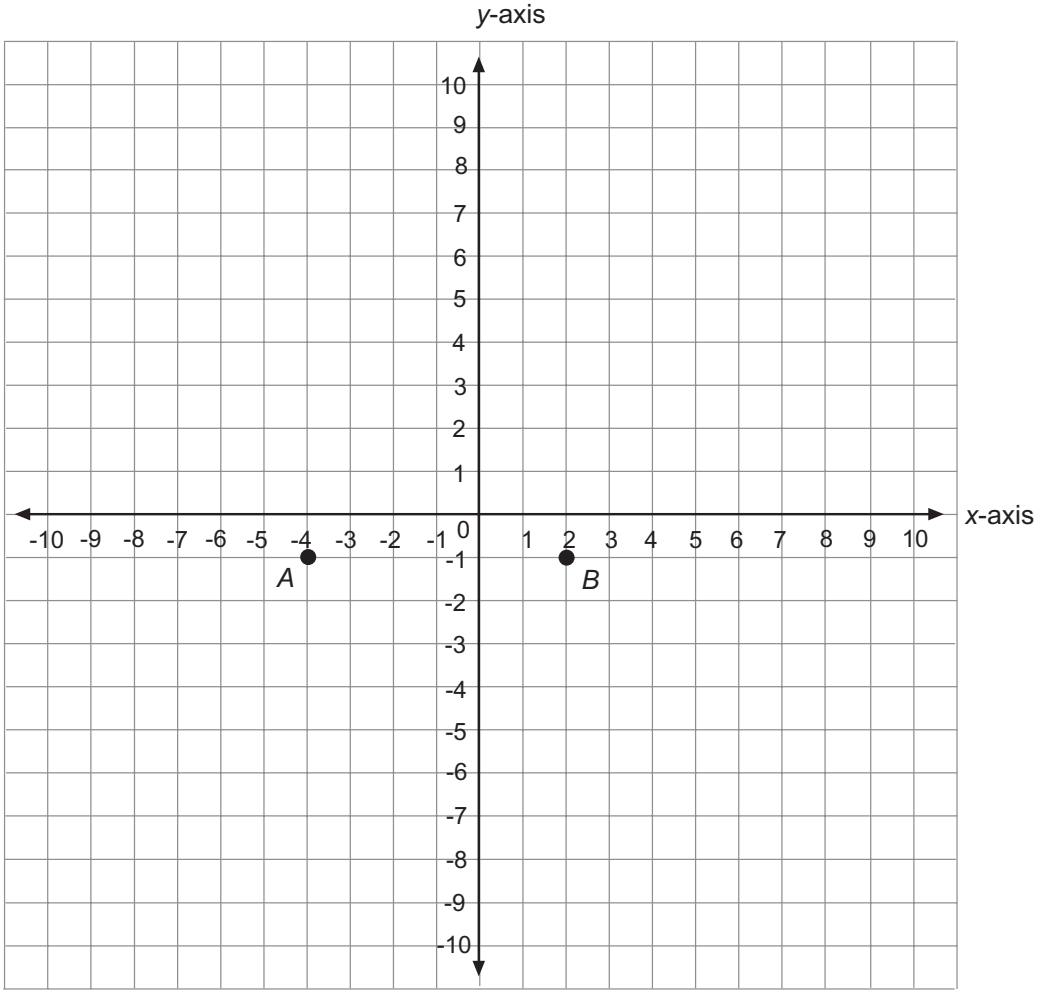
Lesson One Purpose

- Understand geometric concepts such as perpendicularity, parallelism, congruency, reflections, symmetry, and transformations including flips, slides, turns, enlargements, and rotations. (MA.C.2.4.1)
- Using a rectangular coordinate system (graph), apply and algebraically verify properties of two-dimensional figures, including distance, midpoint, slope, parallelism, and perpendicularity. (MA.C.3.4.2)

Distance

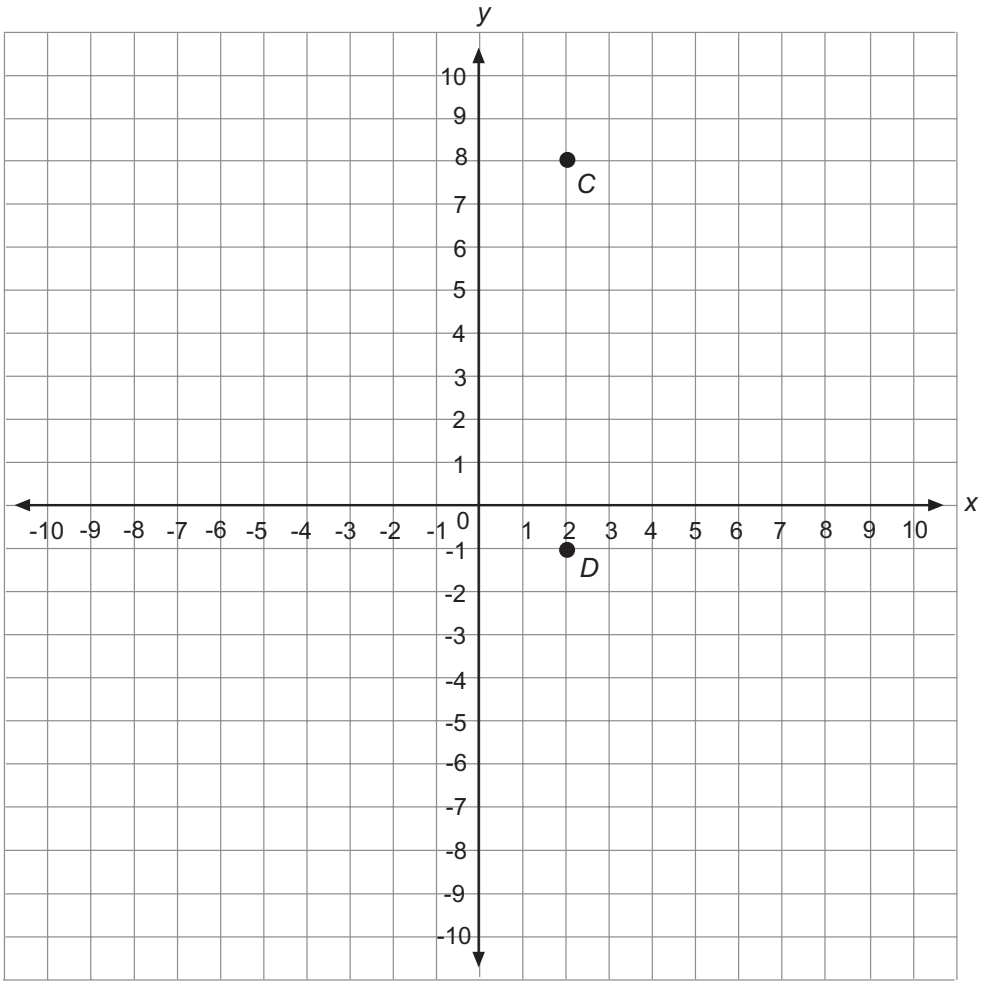
Look at the following **coordinate grids or planes**. The horizontal number line on a *rectangular coordinate system* is the ***x*-axis**. The vertical line on a coordinate system is the ***y*-axis**. We can easily find the **distance** between the given **graphs of the points** below.

Graph of Points *A* and *B*



Because the **points** on the **graph** above are on the same **horizontal (⟷) line**, we can count the spaces from one *point* to the other. So, the *distance* from *A* to *B* is 6.

Graph of Points *C* and *D*



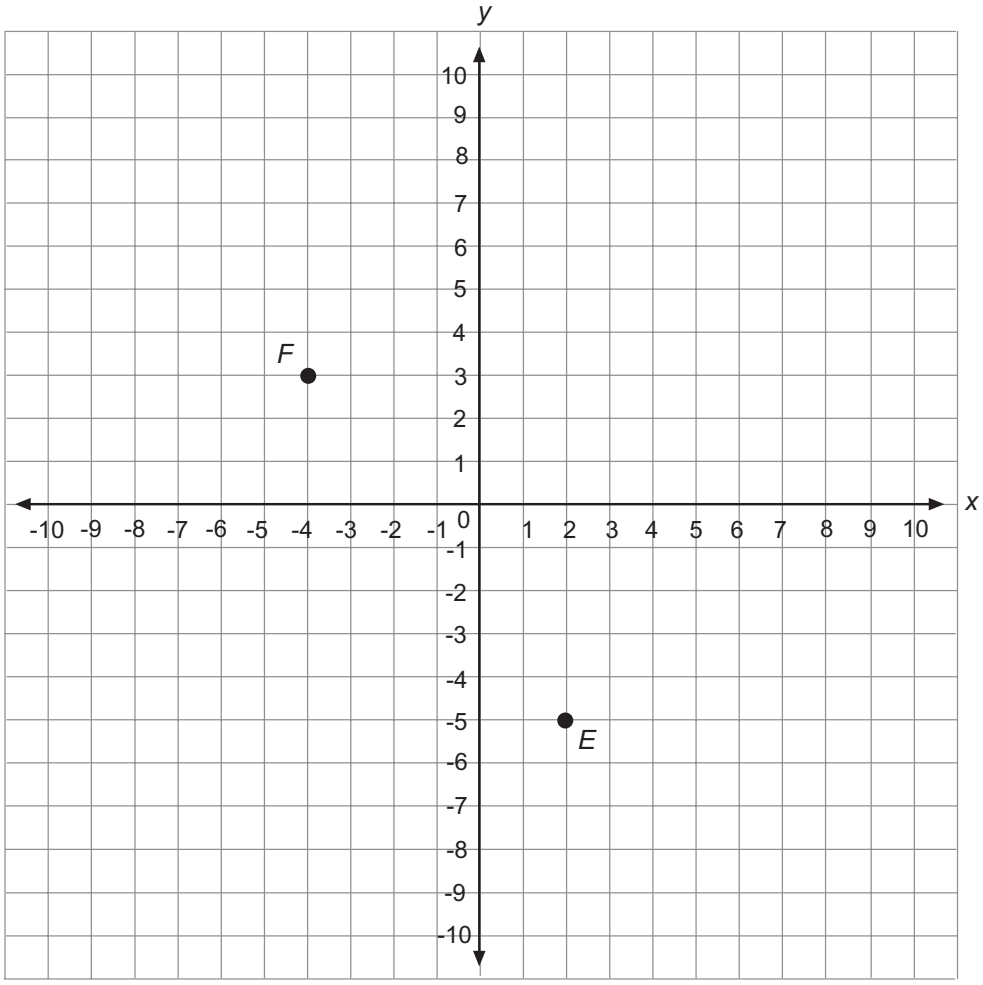
Because the points on the graph above are on the same **vertical (↕)** *line*, we can count the spaces from one point to the other. So, the distance from *C* to *D* is 9.

Remember: Distance is always a **positive number**. Even when you back your car down the driveway, you have covered a *positive* distance. If you get a **negative number**, simply take the **absolute value** of the number.

In many instances, the points we need to find the distance between are not on the same *horizontal* or *vertical* line. Because we would have to count on a **diagonal**, we would not get an accurate measure of the distance between those points. We will examine two methods to determine the distance between any two points.

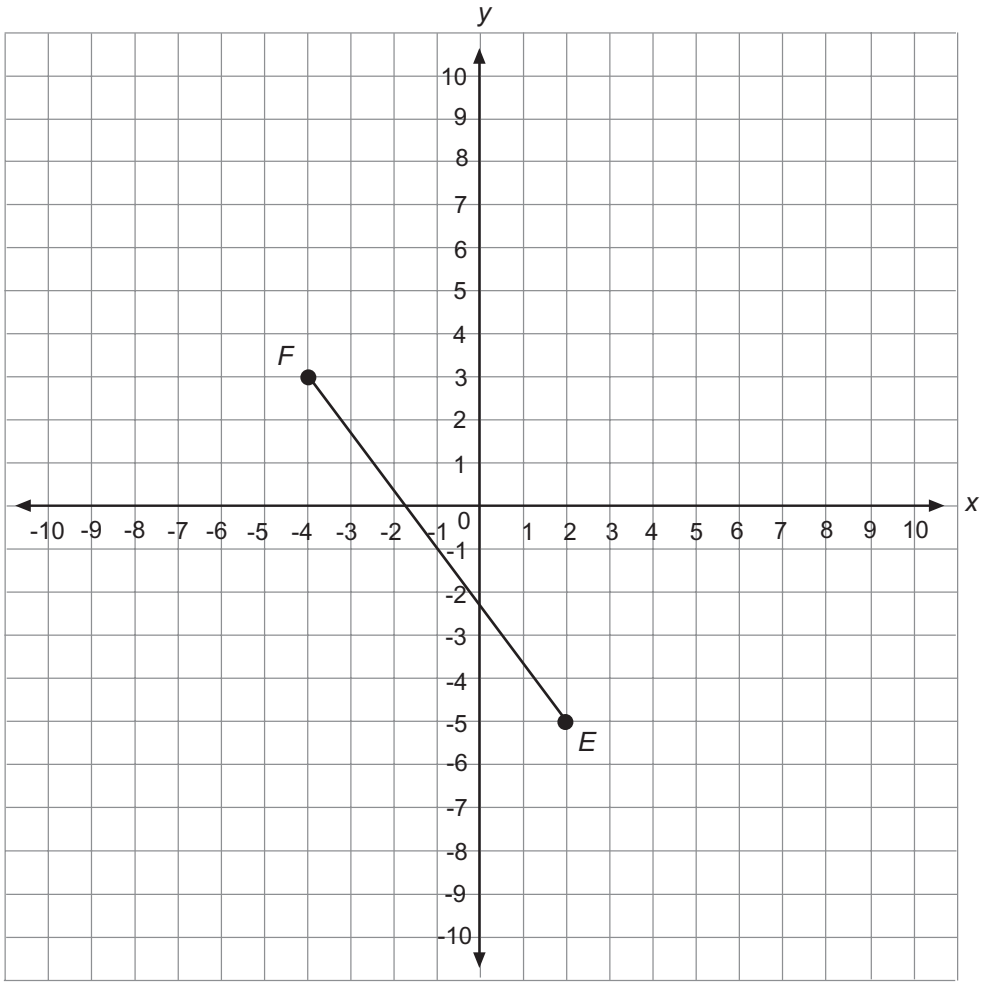
Look at the graph below. We want to find the distance between point *E* (2, -5) and *F* (-4, 3).

Graph of Points *E* and *F*



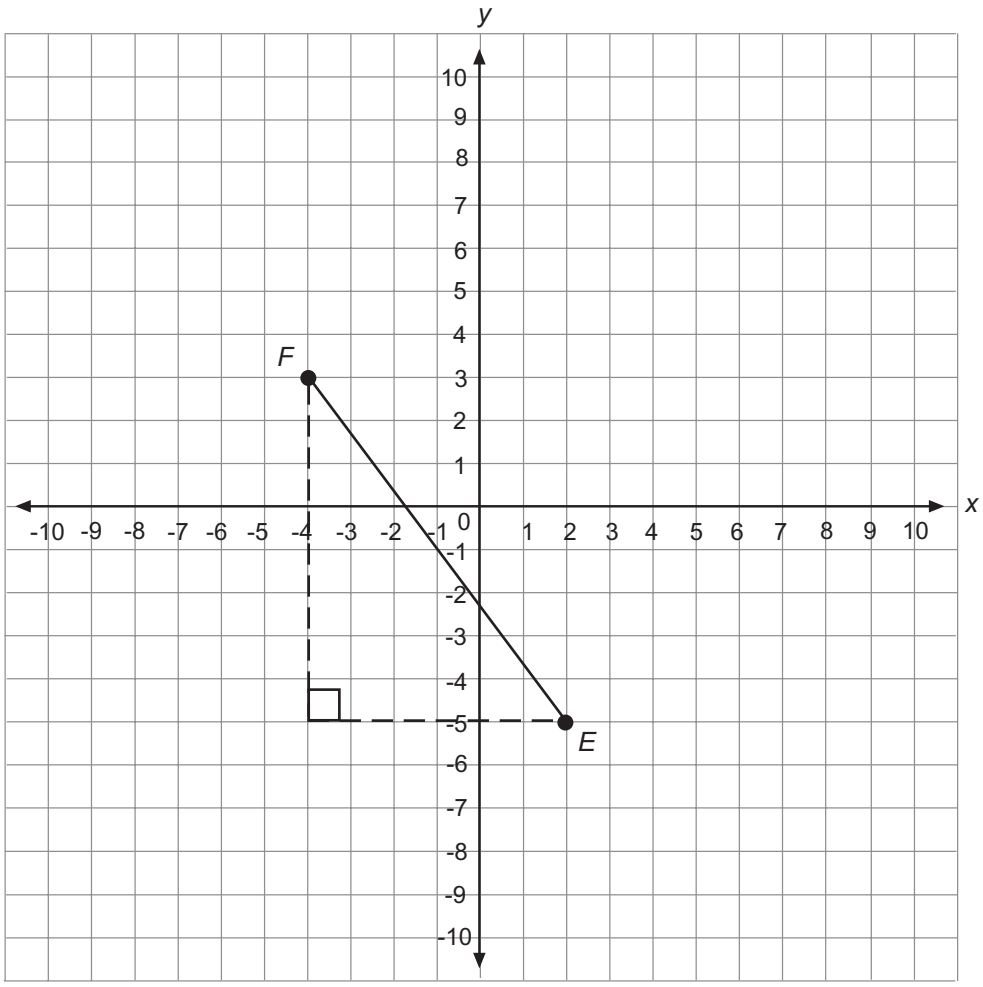
Notice that the distance between *E* and *F* looks like the **hypotenuse** of a **right triangle**.

Graph of Points *E* and *F*



Let's sketch the rest of the **triangle** and see what happens.

Graph of Points *E* and *F*



By completing the sketch of the *triangle*, we see that the result is a *right triangle* with one horizontal **side** and one vertical *side*. We can count to find the **lengths (*l*)** of these two sides, and then use the **Pythagorean theorem** to find the distance from *E* to *F*.

Remember: The *Pythagorean theorem* is the **square** of the *hypotenuse* (c) of a right triangle is equal to the **sum** of the *squares* of the **legs** (a and b), as shown in the **equation** $a^2 + b^2 = c^2$.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 6^2 + 8^2 &= c^2 \\ 36 + 64 &= c^2 \\ 100 &= c^2 \\ \sqrt{100} &= c \\ 10 &= c \end{aligned}$$

Remember:

- The opposite of *squaring* a number is called *finding the* **square root**. For example, the *square root* of 100, or $\sqrt{100}$, is 10.
- The square root of a number is shown by the symbol $\sqrt{\ }$, which is called a **radical sign** or *square root sign*.
- The number underneath is called a **radicand**.
- The **radical** is an **expression** that has a **root**. A *root* is an equal **factor** of a number.
- $\sqrt{100}$ is a **radical expression**. It is a numerical *expression* containing a *radical sign*.

radical sign → $\sqrt{100}$ ← radicand

radical

$$\sqrt{100} = 10 \text{ because } 10^2 = 100$$

$$\sqrt{9} = 3 \text{ because } 3^2 = 9$$

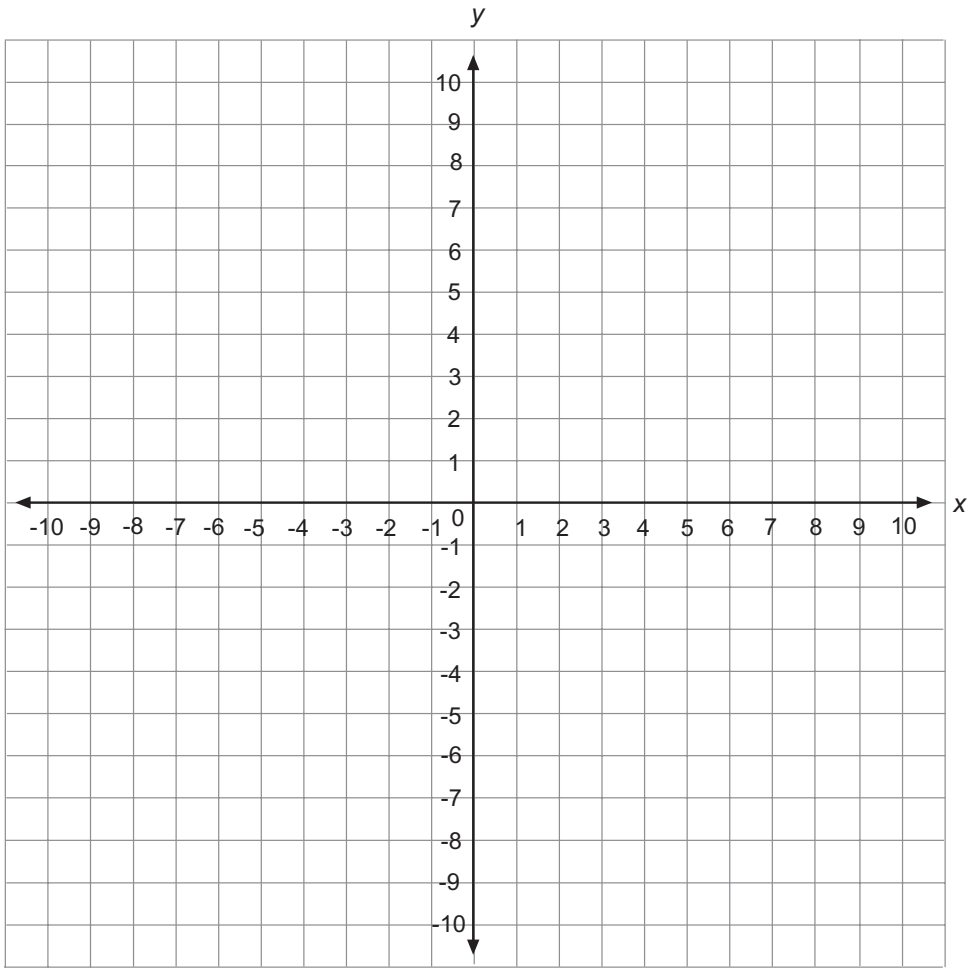
$$\sqrt{121} = 11 \text{ because } 11^2 = 121$$

Practice

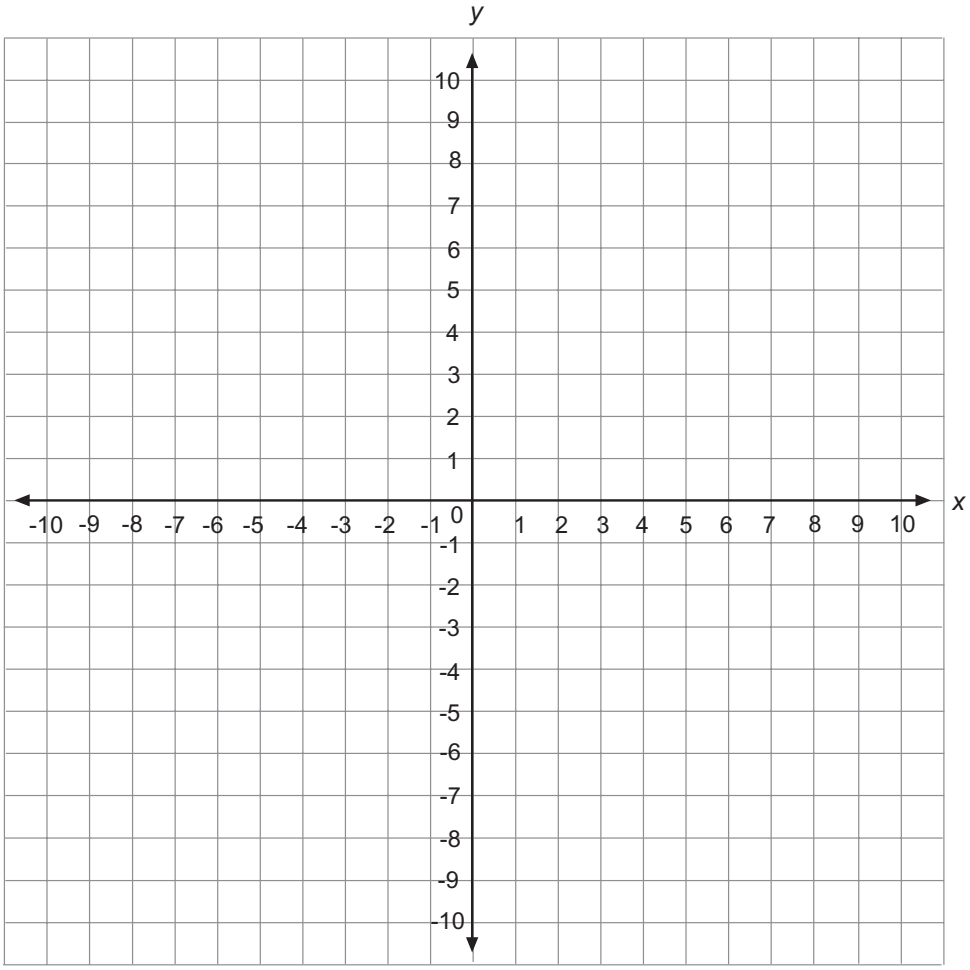
For each of the following:

- *plot the two points*
- *draw the hypotenuse*
- *complete the triangle*
- *use the Pythagorean theorem to find the distance between the given points*
- *show all your work*
- *leave answers in* **simplest radical form**.

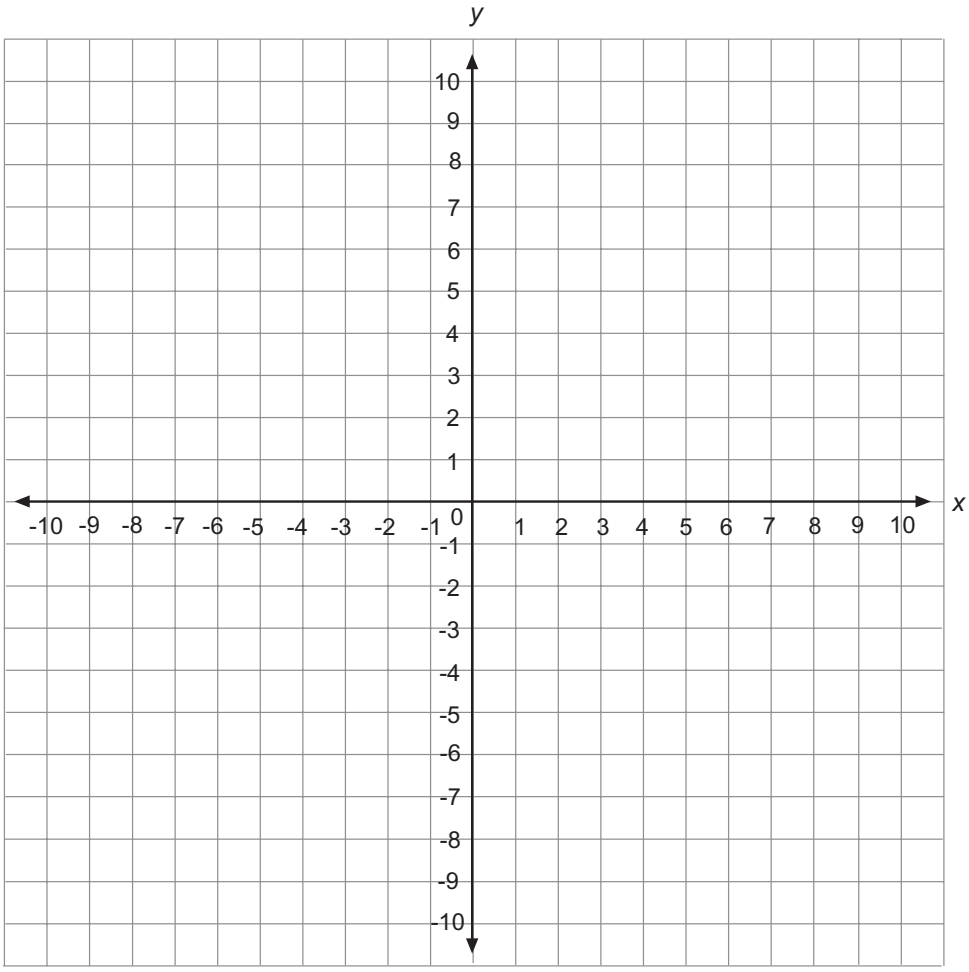
1. (3, 4), (-2, 6)



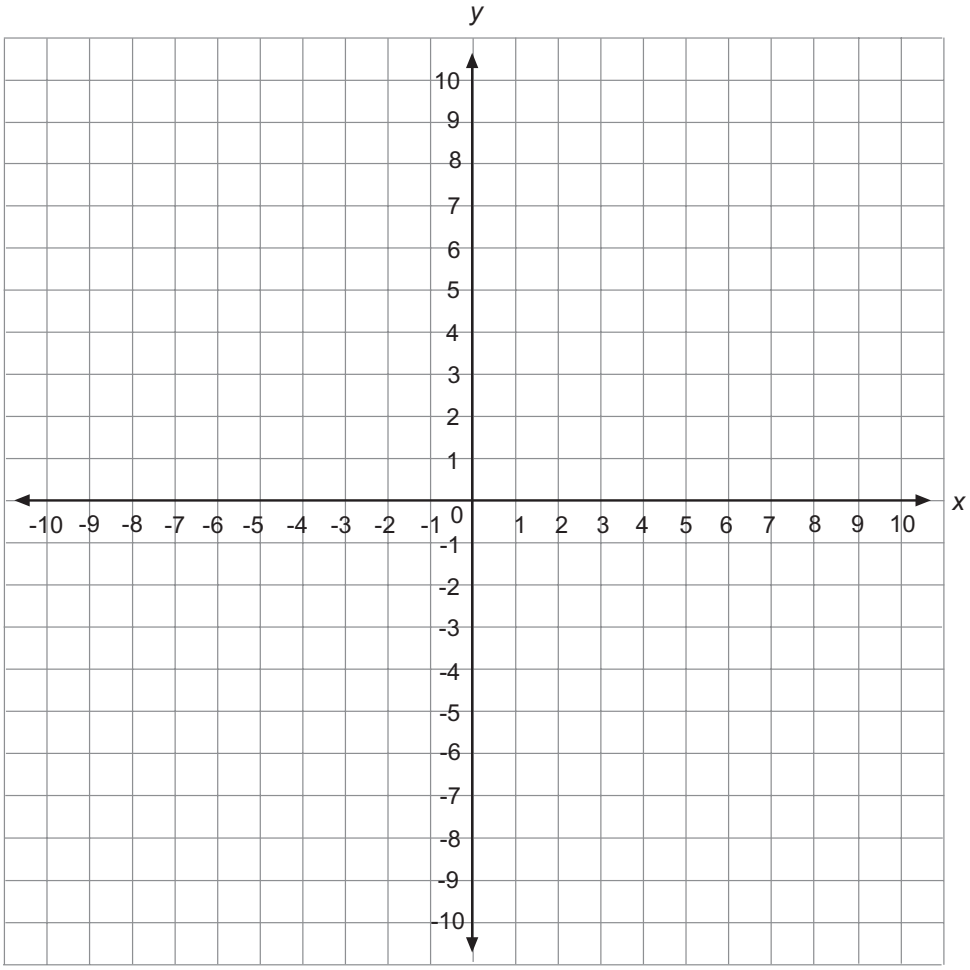
2. (3, -3), (6, 4)



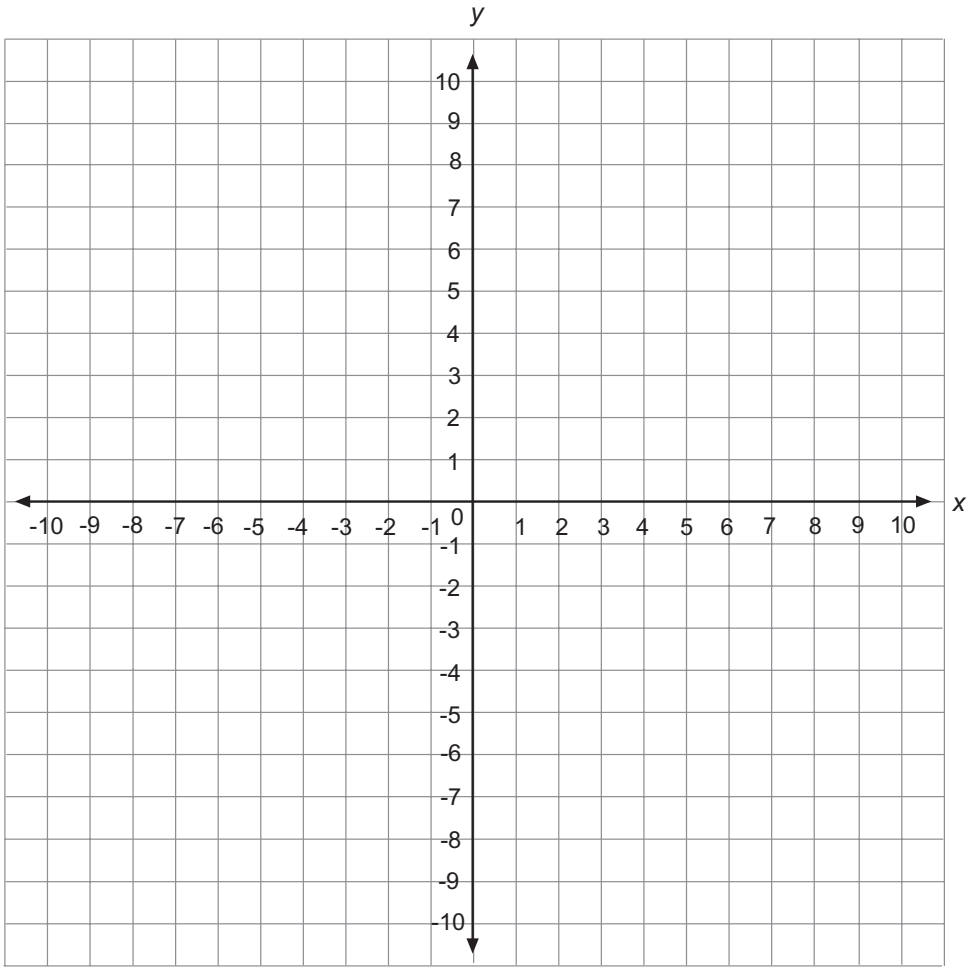
3. (-5, 0), (2, 3)



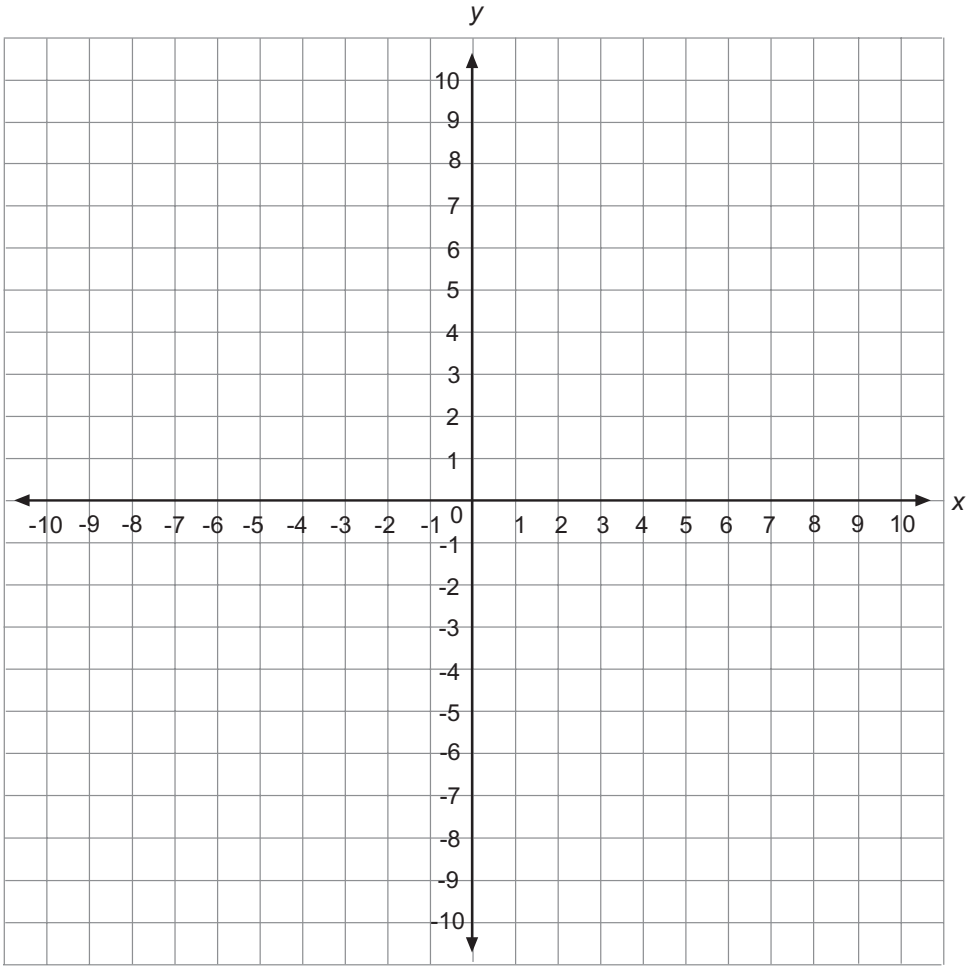
4. (4, -3), (-3, 4)



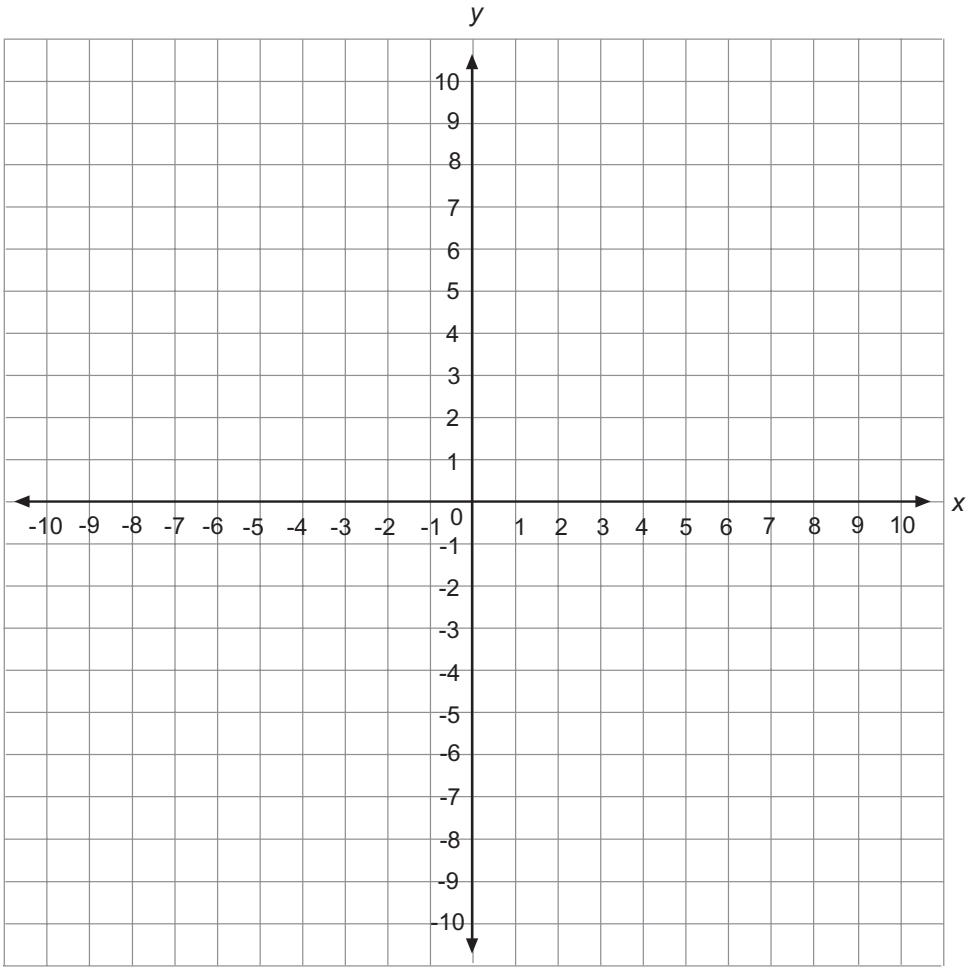
5. (0, 2), (-5, 7)



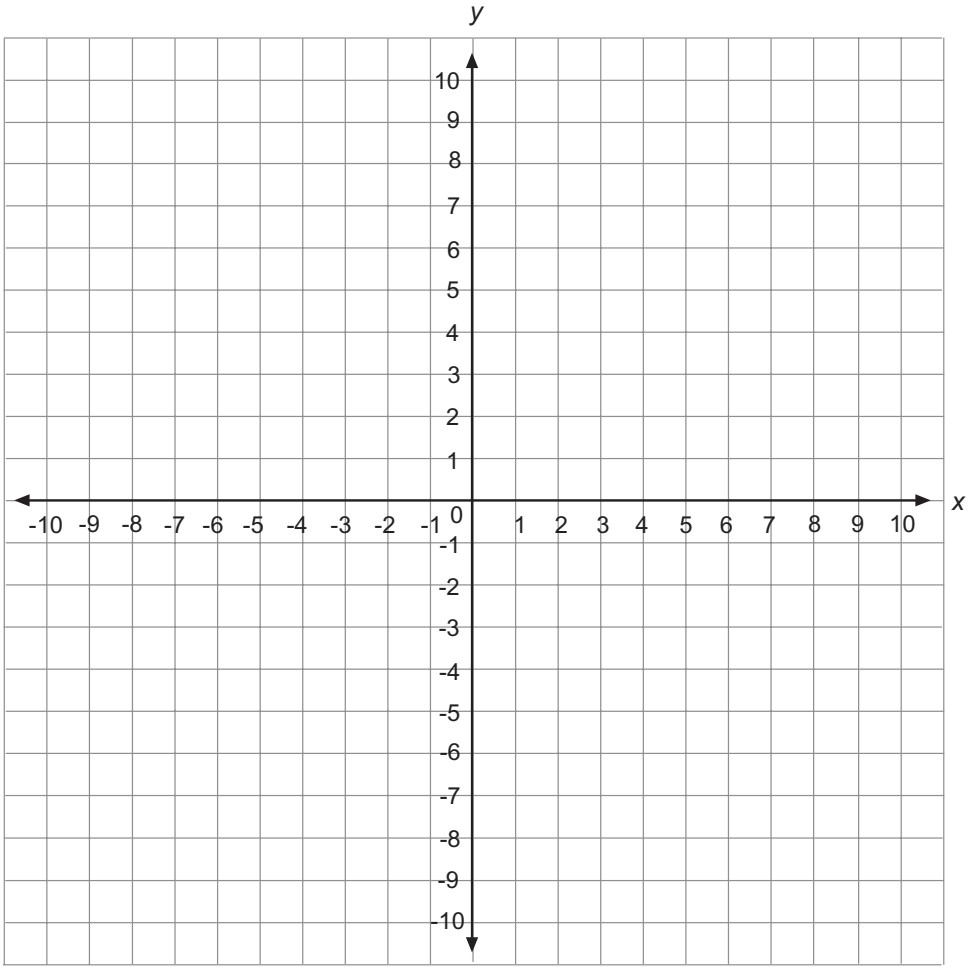
6. (2, 2), (-1, -2)



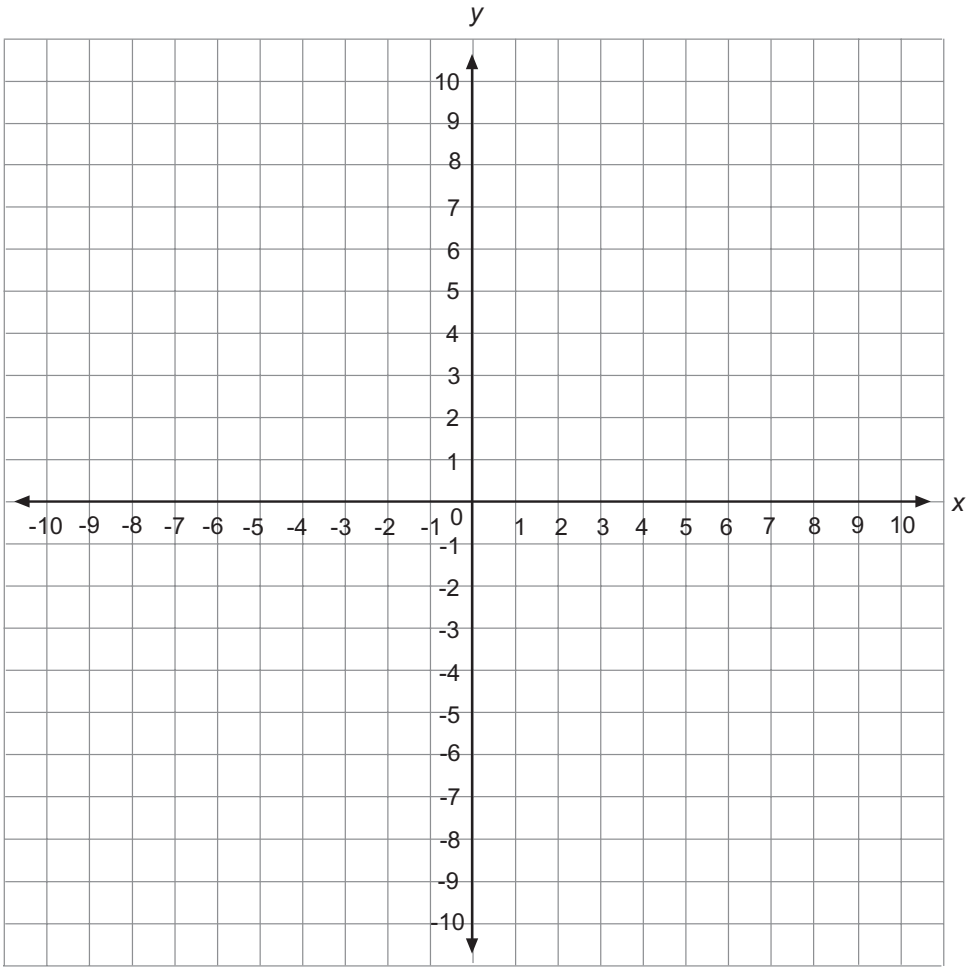
7. (0, 0), (-4, 4)



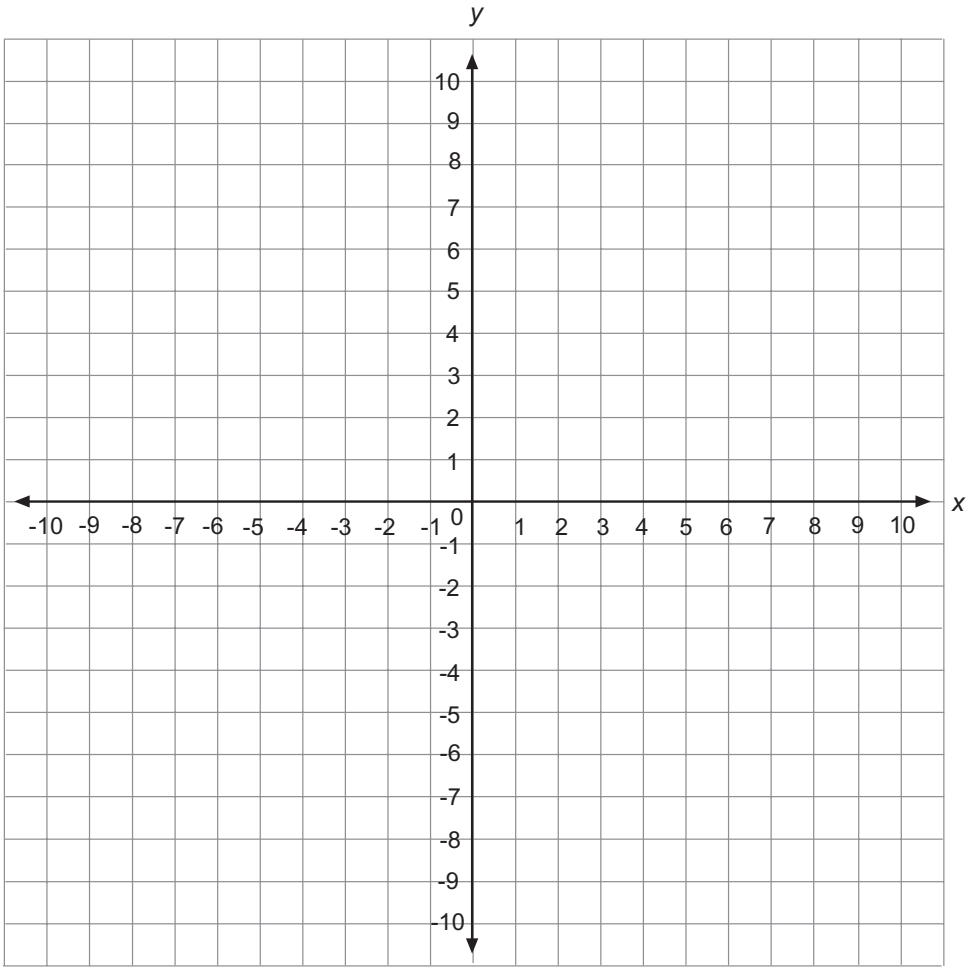
8. (3, 5), (-2, -7)



9. (6, -7), (-2, 8)



10. (-4, 6), (5, -6)



Practice

Use the list below to write the correct term for each definition on the line provided.

absolute value	**horizontal**	**vertical**
coordinate grid or plane	**negative numbers**	***x*-axis**
distance	**positive numbers**	***y*-axis**
graph (of a point)		

_____ 1. parallel to or in the same plane of the horizon

_____ 2. the length of a segment connecting two points

_____ 3. at right angles to the horizon; straight up and down

_____ 4. numbers less than zero

_____ 5. a number's distance from zero (0) on a number line

_____ 6. numbers greater than zero

_____ 7. the vertical number line on a rectangular coordinate system

_____ 8. the point assigned to an ordered pair on a coordinate plane

_____ 9. the horizontal number line on a rectangular coordinate system

_____ 10. a two-dimensional network of horizontal and vertical lines that are parallel and evenly-spaced

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. a one-dimensional measure that is the measurable property of line segments

_____ 2. the longest side of a right triangle; the side opposite the right angle

_____ 3. the square of the hypotenuse (c) of a right triangle is equal to the sum of the square of the legs (a and b)

_____ 4. the edge of a polygon

_____ 5. a polygon with three sides

_____ 6. a triangle with one right angle

_____ 7. the result of adding numbers together

_____ 8. in a right triangle, one of the two sides that form the right angle

_____ 9. the result when a number is multiplied by itself or used as a factor twice

A. hypotenuse

B. leg

C. length (l)

D. Pythagorean theorem

E. right triangle

F. side

G. square (of a number)

H. sum

I. triangle

Using the Distance Formula

Sometimes, it is inconvenient to *graph* when finding the distance. So, another method we often use to find the distance between two points is the distance **formula**.

The distance *formula* is as follows:

distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The little 1s and 2s that are *subscripts* to the xs and ys are meant to show that they come from different **ordered pairs**.

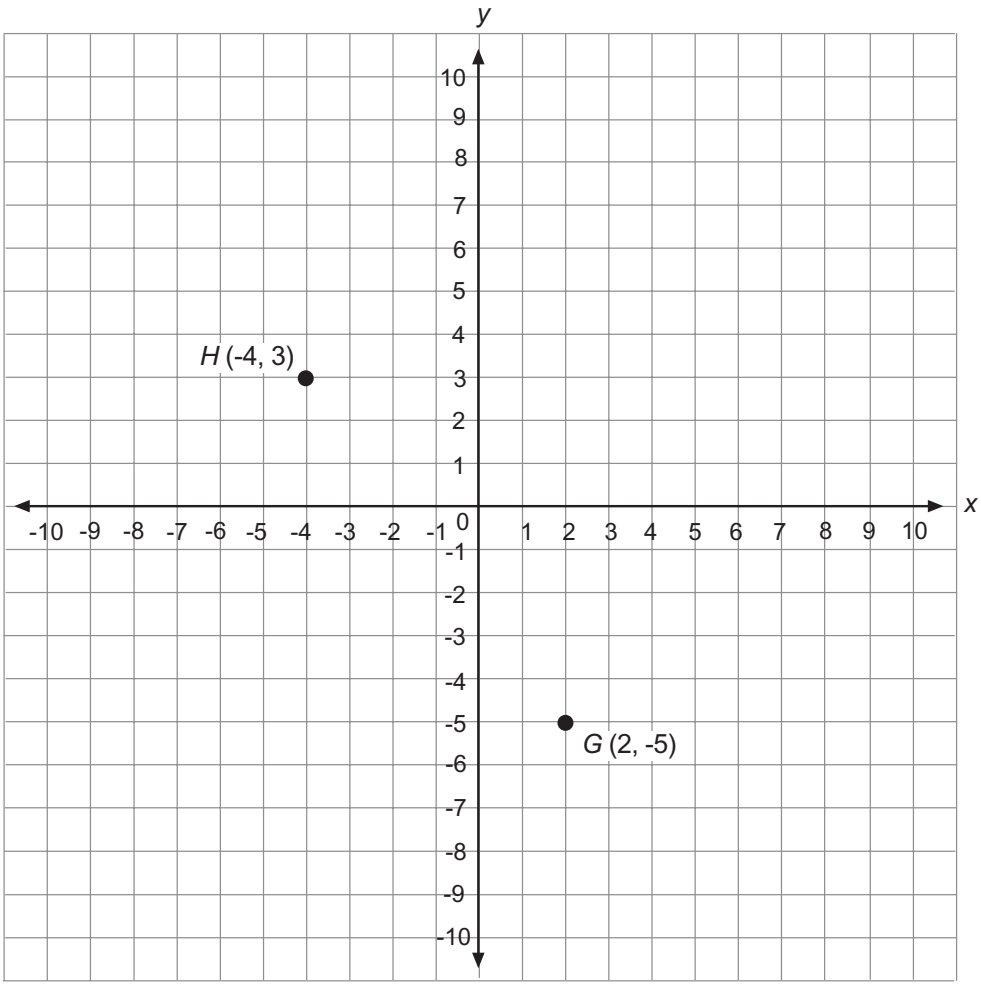
Example

(x_1, y_1) is one *ordered pair* and (x_2, y_2) is another ordered pair.

Note: Be consistent when putting the **values** into the formulas.

Let's look at the same example of G (2, -5) and H (-4, 3), and use the distance formula. See the graph on the following page.

Graph of Points *G* and *H*



$x_1 = 2$
$y_1 = -5$
$x_2 = -4$
$y_2 = 3$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\sqrt{(-4 - 2)^2 + (3 - -5)^2} =$$

$$\sqrt{(-6)^2 + (8)^2} =$$

$$\sqrt{36 + 64} =$$

$$\sqrt{100} =$$

$$10$$

Compare the numbers in the distance formula to the numbers used in the *Pythagorean theorem*.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 6^2 + 8^2 &= c^2 \\ 36 + 64 &= c^2 \\ 100 &= c^2 \\ \sqrt{100} &= c \\ 10 &= c \end{aligned}$$

You should always get the same answer using either method.

Practice

Use the distance formula to solve the following. **Show all your work.** *Leave answers in* **simplest radical form**.

distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1. (3, 4), (-2, 6)

2. (3, -3), (6, 4)

3. (-5, 0), (2, 3)

4. (4, -3), (-3, 4)

5. (0, 2), (-5, 7)

6. (2, 2), (-1, -2)

7. (0, 0), (-4, 4)

8. (3, 5), (-2, -7)

9. (6, -7), (-2, 8)

10. (-4, 6), (5, -6)

✓ **Check yourself:** Compare your answers to the practice on pages 342-351. Do they match? If not, rework until both sets of practice answers match.

Practice

Use your **favorite of the two methods** *shown on pages 354-356. One method uses the* **distance formula** *and the other method uses the* **Pythagorean theorem**. *Find the* **distance between each pair of points** *below using either method. Refer to the examples on pages 354-356 as needed.*

Show all your work. *Leave answers in* **simplest radical form**.

distance formula	**Pythagorean theorem**
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	$a^2 + b^2 = c^2$

1. (0, 0), (-3, 4)

2. (5, -6), (6, -5)

3. (-5, -8), (3, 7)

4. (-2, -8), (0, 0)

5. (6, 6), (-3, -3)

6. (-1, 2), (5, 10)

Lesson Two Purpose

- Using a rectangular coordinate system (graph), apply and algebraically verify properties of two-dimensional figures, including distance, midpoint, slope, parallelism, and perpendicularity. (MA.C.3.4.2)

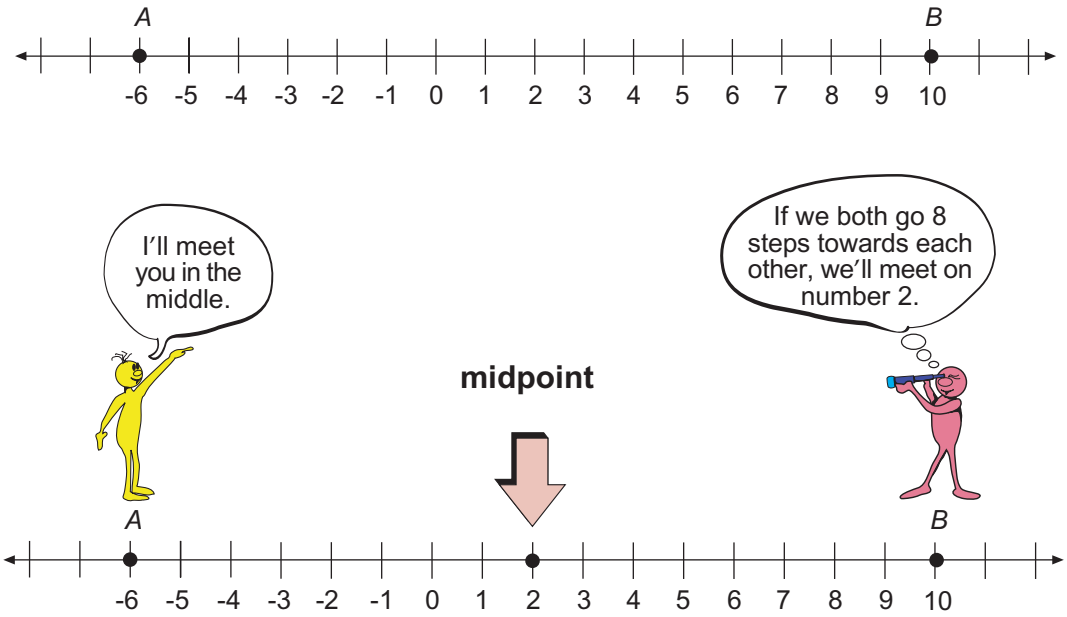
Midpoint

Sometimes it is necessary to find the *point* that is exactly in the middle of two given **endpoints**. We call this the **midpoint (of a line segment)**. What we are actually trying to find are the **coordinates** of that point, which is like the *address* of the point, or its *location* on a graph.

Finding the Midpoint of a Line Segment Using a Number Line

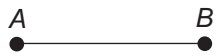
You can find the *midpoint* of a **line segment (—)**, also called a *segment*, in a couple of different ways. One way is to use a **number line**.

On a *number line*, you can find the midpoint of a *line segment* by counting in from both *endpoints* until you reach the middle.



how to use a number line to find the midpoint of a line segment

Remember: If we draw a *line segment* from one point to another, we can call it line segment *AB* or *segment AB*. See a representation of line segment *AB* ($\overline{AB}$) below. The symbol (—) drawn over the two uppercase letters describes a line segment. The symbol has no arrow because the line segment has a definite beginning and end called endpoints. *A* and *B* are endpoints of the line segment *AB* ($\overline{AB}$).



On the other hand, the symbol (⟷) drawn over two uppercase letters describes a line. The symbol has arrows because a line has no definite beginning nor end. *A* and *B* are points on the line *AB* ($\overleftrightarrow{AB}$).

Method One Midpoint Formula

Another way to find the midpoint of a line segment is to use the Method One midpoint formula below. To do this, add the two endpoints together and divide by two.

Method One midpoint formula

$$\frac{a+b}{2}$$

$$\frac{a+b}{2} =$$

$$\frac{-6+10}{2} =$$

$$\frac{4}{2} =$$

$$2$$

Therefore, for points *A* and *B* on the *number line*, the midpoint is

$$\frac{-6+10}{2} = \frac{4}{2} = 2.$$

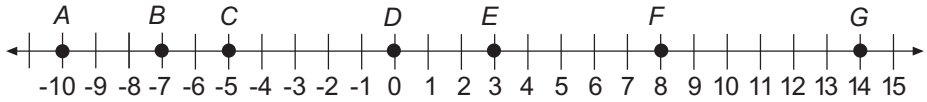
Practice

Find the **coordinate of the midpoint** *for each* **pair of points** *on the* **number line** *below. Use either of the methods below from pages 362-363.*

- **Use the number line and count in from both endpoints** *of a line segment until you reach the middle to determine the midpoint.*
- **Use the Method One midpoint formula** *and add the two endpoints together, then divide by two.* **Show all your work.**

Method One midpoint formula

$$\frac{a+b}{2}$$



Refer to previous pages as needed.

1. *A* and *C*

2. *B* and *E*

3. A and E

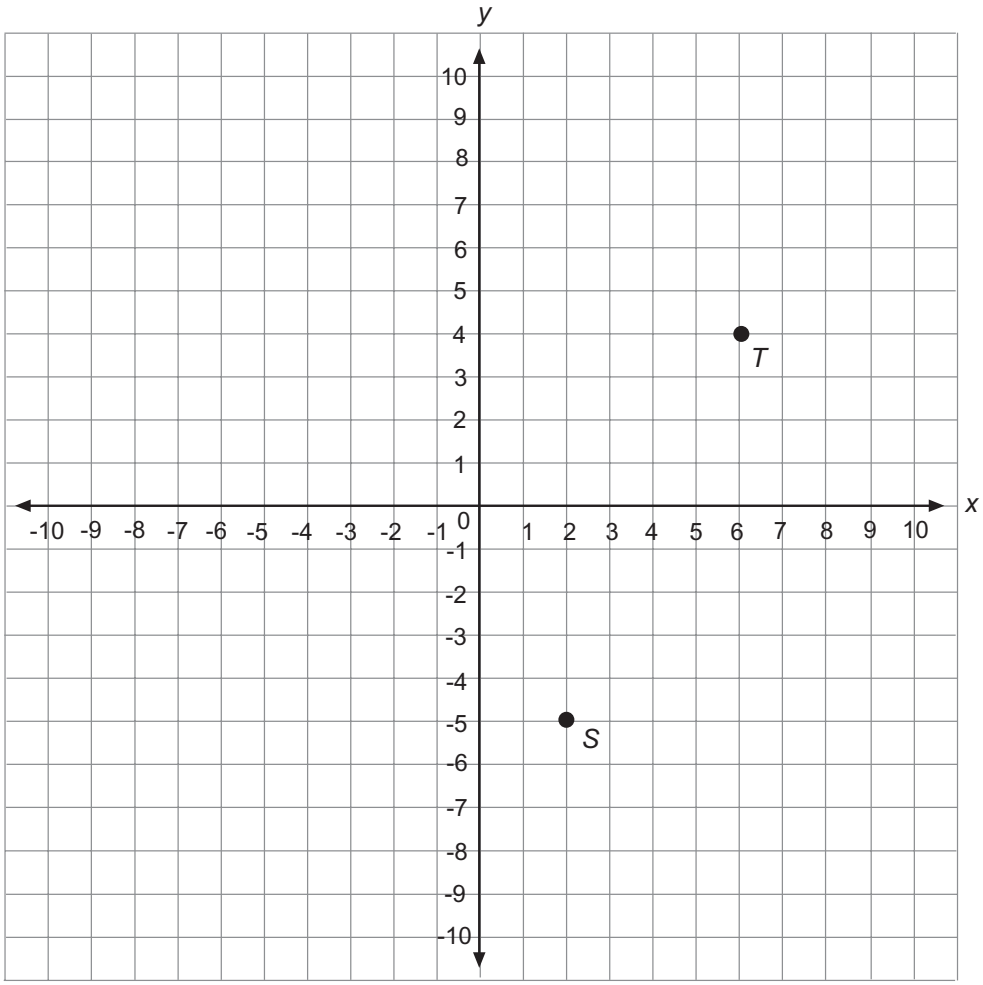
4. D and G

5. A and G

Method Two Midpoint Formula

Do you think the process may change a bit when we try to find the midpoint of points *S* and *T* as seen on the graph below?

Graph of Points *S* and *T*



When the points are on a **coordinate plane**, or the plane containing the *x*- and *y*-axes, we have to think in two dimensions to find the *coordinates* of the midpoint. The midpoint will have an ***x*-coordinate** and a ***y*-coordinate** (x, y). To find the midpoint on a coordinate plane, we simply use the Method Two midpoint formula twice—once to find the *x-coordinate* and again to find the *y-coordinate*.

Method Two midpoint formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Let's see how this works.

We see that point S has coordinates (2, -5), and T is located at (6, 4). Use the Method Two midpoint formula to find the exact location of the midpoint of $\overline{ST}$.

$$\text{midpoint of } \overline{ST} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) =$$

find the *average* of the *x*-values, then the average of the *y*-values

$$\left(\frac{2 + 6}{2}, \frac{-5 + 4}{2}\right) =$$

add the *x*'s then the *y*'s

$$\left(\frac{8}{2}, \frac{-1}{2}\right) =$$

now **simplify the fraction**

$$\left(4, \frac{-1}{2}\right) =$$

Practice

Find the **midpoint** *of the* **coordinates** *for each* **segment** *whose endpoints are given. Use the* **Method Two midpoint formula** *below.* **Show all your work.** *Refer to pages 366-367 as needed.*

Method Two midpoint formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

1. (2, 8), (-4, 2)

2. (0, 0), (-3, -4)

3. (1, 2), (4, 3)

4. (-3, -5), (9, 0)

5. (-4, 6), (3, 3)

6. (5, -6), (-5, 6)

7. (6, 6), (-4, -4)

8. (5, 5), (-5, -5)

9. (8, -4), (10, 9)

10. (6, 8), (-3, 5)

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. a portion of a line that consists of two defined endpoints and all points in between

_____ 2. the plane containing the x- and y-axes

_____ 3. write fraction in lowest terms or simplest form

_____ 4. the second number of an ordered pair

_____ 5. the number paired with a point on the number line

_____ 6. the first number of an ordered pair

_____ 7. the point on a line segment equidistant from the endpoints

_____ 8. a line on which ordered numbers can be written or visualized

A. coordinate

B. coordinate plane

C. line segment (—)

D. midpoint (of a line segment)

E. number line

F. simplify a fraction

G. x-coordinate

H. y-coordinate

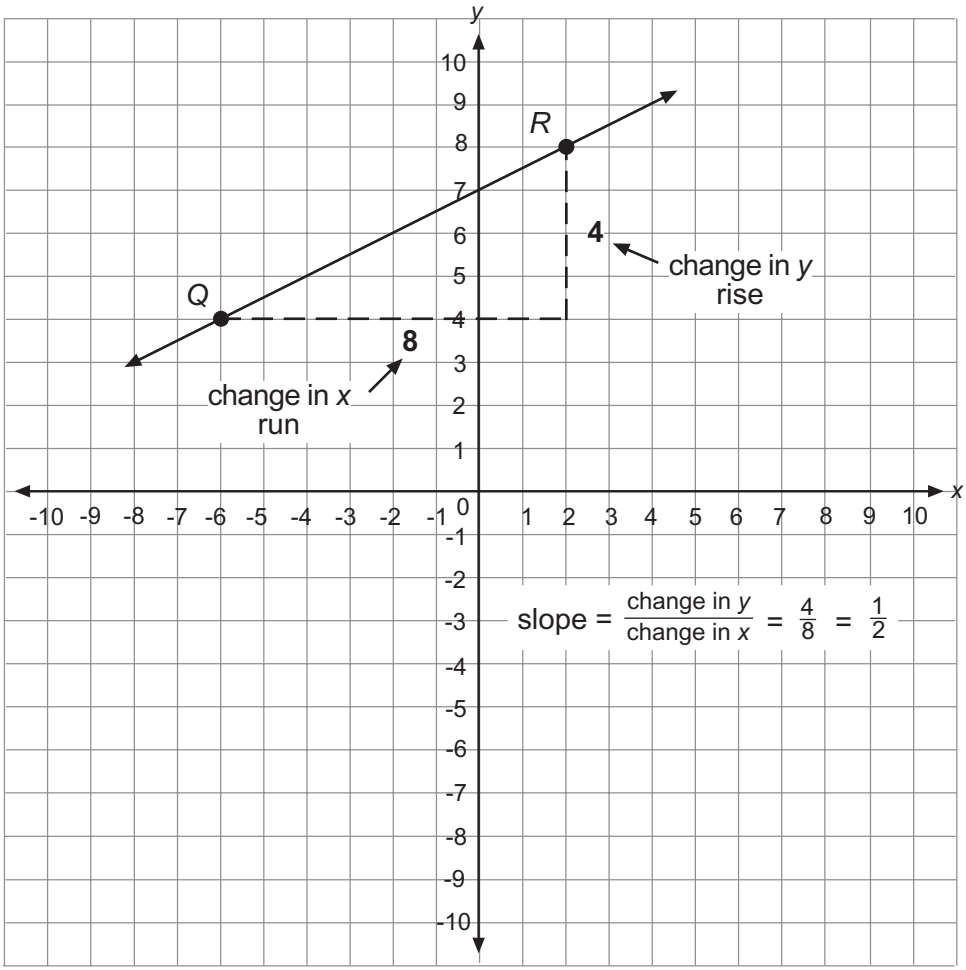
Lesson Three Purpose

- Using a rectangular coordinate system (graph), apply and algebraically verify properties of two-dimensional figures, including distance, midpoint, slope, parallelism, and perpendicularity. (MA.C.3.4.2)

Slope

Slope can be thought of as the slant of a line. It is often defined as $\frac{\text{rise}}{\text{run}}$, which means the change in the *y*-values (**rise**) on the *vertical* axis, divided by the change in the *x*-values (**run**) on the *horizontal* axis. In the figure below we can count to find the *slope* between points *Q* (-6, 4) and *R* (2, 8).

Graph of Points *Q* and *R*



slope of a line

However, we can also use the *slope formula* to determine the slope of a line without having to see a graph of the two points of the line.

slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Remember: m is always used to represent slope.

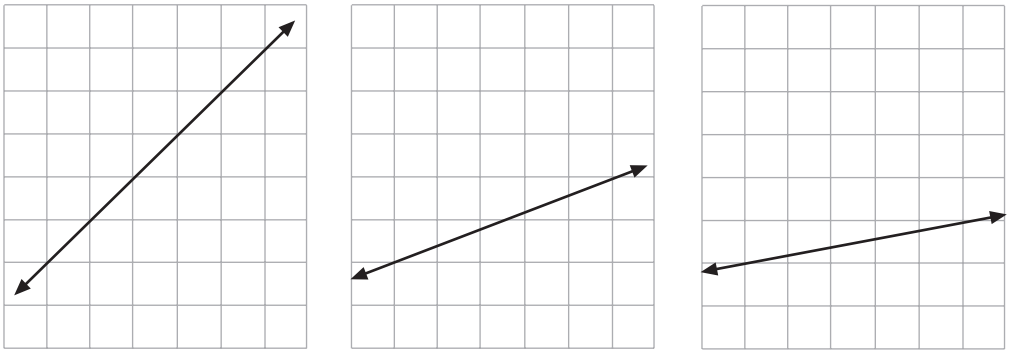
However, we must know the coordinates of two points on a line so that we can use the formula. Refer to points Q and R on the previous page. The coordinates of Q are (-6, 4) and the coordinates of R are (2, 8). Let's see how this works in the slope formula.

$x_1 = -6$
$x_2 = 2$
$y_1 = 4$
$y_2 = 8$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 4}{2 - -6} = \frac{4}{8} = \frac{1}{2}$$

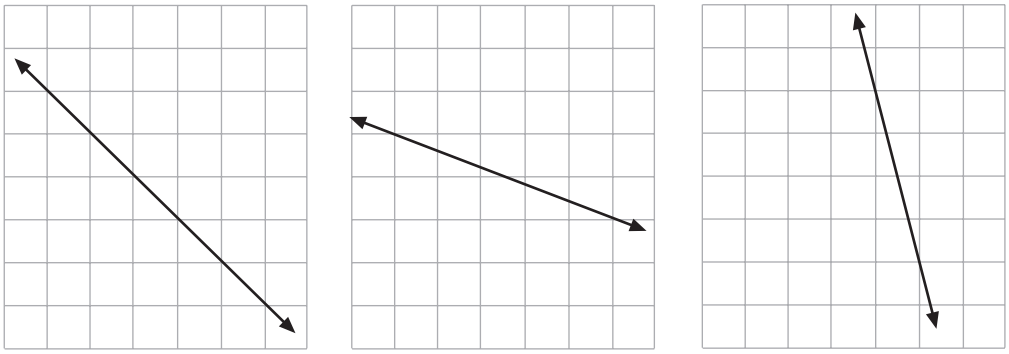
When the slope of a line is *positive*, the line will *rise*.

Examples

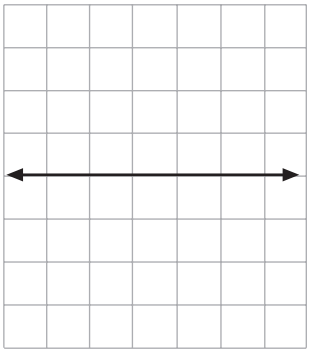


When the slope of a line is *negative*, the line will *fall*.

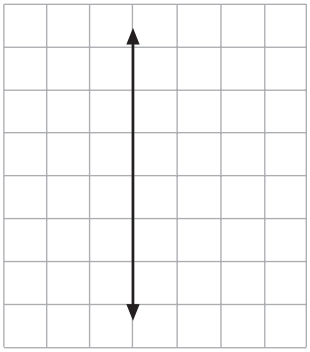
Examples



When the slope has a zero in the **numerator** ($\frac{0}{x}$), the line will be *horizontal* and have a slope of 0.



When the slope has a zero in the **denominator** ($\frac{y}{0}$), the line will be *vertical* and have *no* slope at all. We sometimes say that the slope of a vertical line is *undefined*.



Practice

Use the **slope formula** *below to find the slope of each* **line passing through points** *listed below.* **Simplify the answer.** *Then determine whether the* **line** *is* **rising**, **falling**, **horizontal**, *or* **vertical**. *Write the answer on the line provided.* **Show all your work.**

slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Remember:

$\frac{0}{x}$ = a line that is horizontal and has a **zero** (0) **slope**

$\frac{y}{0}$ = a line that is vertical and has **no slope**

_____ 1. (2, 8), (-4, 2)

_____ 2. (0, 0), (-3, -4)

_____ 3. (1, 2), (4, 3)

_____ 4. (3, -6), (3, 4)

_____ 5. (-3, -5), (9, 0)

_____ 6. (-4, 6), (3, 3)

_____ 7. (4, 2), (-5, 2)

_____ 8. (5, -6), (-5, 6)

_____ 9. (6, 6), (-4, -4)

_____ 10. (6, 7), (6, -4)

_____ 11. (5, 5), (-5, -5)

_____ 12. (0, 4), (0, 9)

_____ 13. (8, -4), (10, -9)

_____ 14. (6, 5), (-3, 8)

_____ 15. (4, 5), (8, 16)

Slope-Intercept Form

Recall that you can also tell the slope of a line by examining the *equation* of that line. If a **linear equation** is written in **slope-intercept form**,

slope-intercept form
$y = mx + b$

we can determine the slope of the line by finding the value of m in the equation.

Look at the following examples:

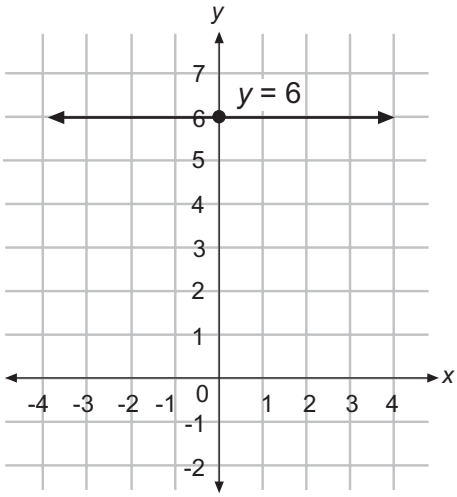
$y = \boldsymbol{m}x + b$

$y = \mathbf{2}x + 4$ slope is **2**

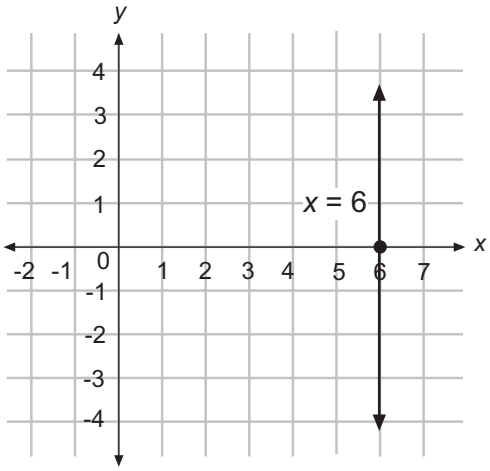
$y = \mathbf{-5}x + 6$ slope is **-5**

$y = \frac{2}{3}x - 7$ slope is $\frac{2}{3}$

Note: If the equation is $y = 6$, the line is *horizontal* and has *zero* slope. If the equation is $x = 6$, the line is *vertical* and has *no* slope. See examples of each below.



y = 6
line is horizontal
zero slope



x = 6
line is vertical
no slope

Practice

Give the **slopes** *for the* **lines** *whose equations are given below.*

Remember:

- The slope-intercept form is $y = mx + b$ and m is the **slope of the line**.
- If y = one number and no **variables**, the line is horizontal and has **zero slope**.
- If x = one number and no variables, the line is vertical and has **no slope**.

1. $y = 5x + 7$

2. $y = -3x - 9$

3. $y = \frac{5}{7}x + 11$

4. $x = -7$

5. $y = -\frac{5}{3}x - 6$

6. $y = 8$

7. $x = 2$

8. $y = -5x$

9. $y = x + 2$

10. $y = -x - 8$

Practice

Use the list below to write the correct term for each definition on the line provided.

denominator	**rise**	**slope**
linear equation	**run**	**slope-intercept form**
numerator		

_____ 1. the vertical change on the graph between two points

_____ 2. a form of a linear equation, $y = mx + b$, where m is the slope of the line and b is the y-intercept

_____ 3. the ratio of change in the vertical axis (y-axis) to each unit change in the horizontal axis (x-axis) in the form $\frac{\text{rise}}{\text{run}}$; the constant, m, in the linear equation for the slope-intercept form $y = mx + b$

_____ 4. the top number of a fraction, indicating the number of equal parts being considered

_____ 5. the bottom number of a fraction, indicating the number of equal parts a whole was divided into

_____ 6. the horizontal change on a graph between two points

_____ 7. an equation whose graph in a coordinate plane is a straight line; an algebraic equation in which the variable quantity or quantities are raised to the zero or first power only and the graph is a straight line

Lesson Four Purpose

- Understand geometric concepts such as perpendicularity, parallelism, congruency, reflections, symmetry, and transformations including flips, slides, turns, enlargements, and rotations. (MA.C.2.4.1)
- Using a rectangular coordinate system (graph), apply and algebraically verify properties of two-dimensional figures, including distance, midpoint, slope, parallelism, and perpendicularity. (MA.C.3.4.2)

Parallel and Perpendicular Lines

When two lines are on the same coordinate plane, there are two possibilities. Either the two lines are **parallel** to each other or they **intersect** each other.

If two lines are *parallel* to each other, we can say that the lines are always the same distance apart and will never *intersect*. This happens when the two lines have the same *slant*. In other words, two **parallel lines** have *equal* slopes.

For example, the two lines, $y = \mathbf{5}x + 13$ and $y = \mathbf{5}x - 6$ are parallel because in each line, m has a value of **5**.

If two lines intersect, they cross each other at some point. You may not see that point where they cross on the particular picture, but remember that lines extend forever and their slopes may be such that they will eventually cross. If the two lines intersect at a **right angle** or at 90 **degrees** (°), they are **perpendicular (⊥)**. They have two lines that intersect to form *right angles*. Keep in mind that when this happens, their slopes will be negative **reciprocals** of each other.

A line whose equation is $y = \frac{3}{2}x - 5$ is *perpendicular* to a line whose equation is $y = -\frac{2}{3}x + 6$. Notice that their slopes are $\frac{2}{3}$ and $-\frac{3}{2}$.

Note: If you multiply the slopes of two **perpendicular lines** together, the **product** will be -1, unless one of the lines was vertical.

Practice

Use the slope formula below to find the **slopes** *of* $\overleftrightarrow{AB}$ *and* $\overleftrightarrow{CD}$. *Then* **multiply the slopes** *to determine if they are* **parallel**, **perpendicular**, *or* **neither**. **Show all your work.** *Write the answer on the line provided.* **The first one has been done for you.**

slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Remember:

- If slopes are **equal**, the lines are **parallel**.
- If slopes are **negative reciprocals**, the lines are **perpendicular**.

parallel

1. *A* (3, 2), *B* (-5, 6), *C* (-4, 1), *D* (-2, 0)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \qquad m = y_2 - y_1 =$$

$$\frac{6-2}{-5-3} = \qquad \frac{0-1}{-2--4} =$$

$$\frac{6+-2}{-5+-3} = \qquad \frac{0+-1}{-2+{}^{+}4} =$$

$$\frac{4}{-8} = \qquad -\frac{1}{2}$$

$$-\frac{1}{2} =$$

The slopes are equal; therefore, the lines are *parallel*.

_____ 2. A (5, 7), B (0, 4), C (2, -6), D (-3, 7)

_____ 3. A (2, 4), B (-6, -6), C (3, -3), D (-1, -8)

_____ 4. A (0, 5), B (3, 5), C (6, 7), D (6, -3)

Remember:

$\frac{0}{x}$ = a line that is horizontal and has a zero (0) slope

$\frac{y}{0}$ = a line that is vertical and has no slope

_____ 5. A (3, 8), B (4, 5), C (0, 0), D (6, -4)

_____ 6. A (4, 4), B (-4, -4), C (-4, 4), D (4, -4)

_____ 7. A (-2, -2), B (2, 4), C (1, 6), D (-1, 3)

_____ 8. A (8, -8), B (0, -6), C (3, 13), D (-3, -11)

Practice

Put equations in **slope-intercept form**. **Show all your work.** *Determine if the following* **lines** *are* **parallel**, **perpendicular**, *or* **neither**. *Write the answer on the line provided.*

slope-intercept form
$y = mx + b$

_____ 1. $y = -3x + 7$
$y = 3x - 6$

_____ 2. $y = x + 6$
$y = -x + 6$

_____ 3. $y = \frac{1}{3}x + 7$
$y = \frac{1}{3}x - 7$

_____ 4. $y = 5x - 9$
$y = 5x - 4$

_____ 5. $y = 3x - 4$
$y = 4x + 3$

_____ 6. $x = 6$
$y = -1$

_____ 7. $2x + 3y = 5$
$3x - 2y = 7$

Practice

Use the list below to complete the following statements.

distance	**line segment**	**perpendicular**
horizontal	**midpoint**	**slope**
hypotenuse	**parallel**	**vertical**

1. The slant or _____ of a line is defined as $\frac{\text{rise}}{\text{run}}$.

2. A line that has no slope is called a _____ line.

3. The _____ between two points is the length of the segment that connects the two points.

4. The _____ is the segment in a right triangle that is opposite the right angle.

5. Lines that are in the same plane and do not intersect are called _____ lines.

6. A line that has zero slope is a _____ line.

7. The point that is located exactly half way between two endpoints of a line segment is called the _____ of a line segment.

8. If two lines intersect to form right angles, they are _____ lines.

9. The figure that contains two defined endpoints and all the points in between is called a _____ .

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. the square of the hypotenuse (c) of a right triangle is equal to the sum of the square of the legs (a and b), as shown in the equation $c^2 = a^2 + b^2$

_____ 2. two lines, two line segments, or two planes that intersect to form a right angle

_____ 3. an angle whose measure is exactly 90°

_____ 4. two lines in the same plane that are a constant distance apart; lines with equal slopes

_____ 5. two numbers whose product is 1; also called *multiplicative inverses*

_____ 6. to meet or cross at one point

_____ 7. a way of expressing a relationship using variables or symbols that represent numbers

_____ 8. the result of multiplying numbers together

A. formula

B. intersect

C. parallel lines

D. perpendicular (⊥)

E. product

F. Pythagorean theorem

G. reciprocals

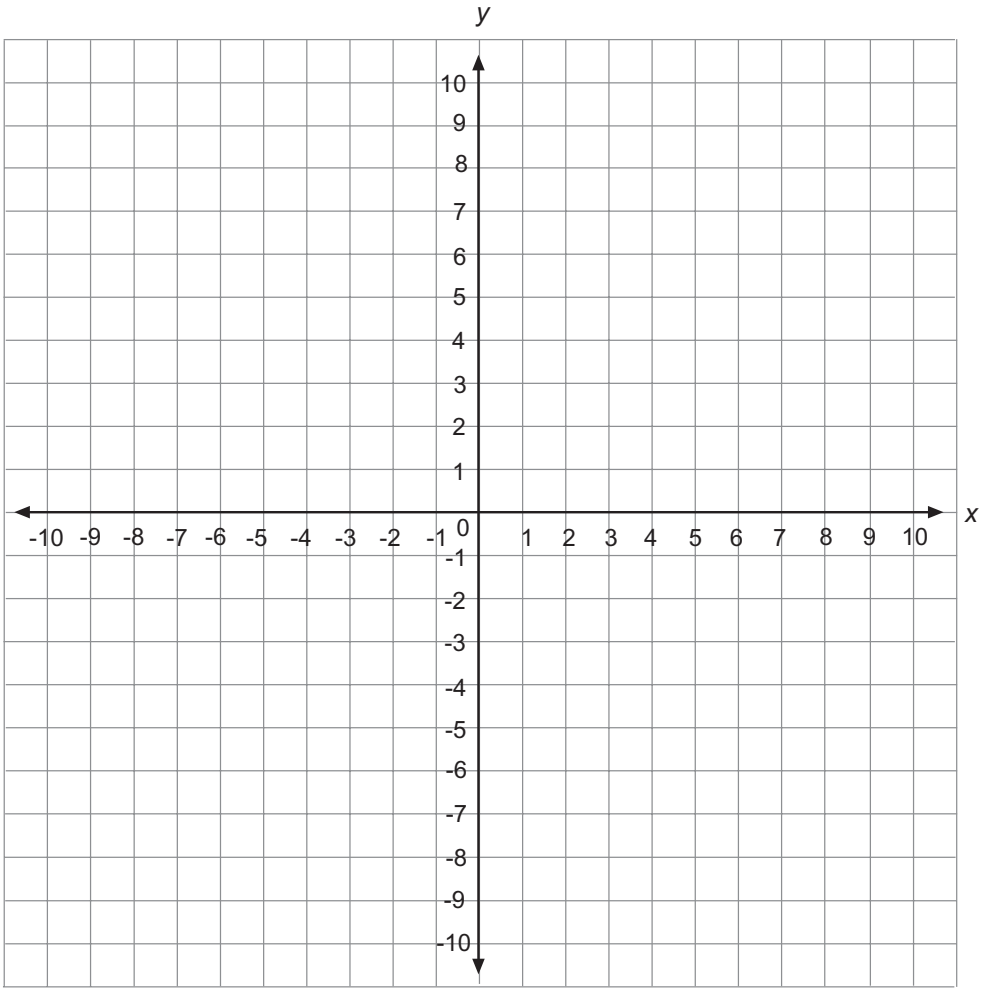
H. right angle

Unit Review

Solve the following.

1. Plot points (3, -2) and (-6, 4). Draw a triangle and use the Pythagorean theorem below to find the distance between the two points.

Pythagorean theorem
$a^2 + b^2 = c^2$



2. Use the distance formula below to find the distance between (-2, 4) and (7, -3).

distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

3. Use either the method above or the Pythagorean theorem below to find the distance between (5, 1) and (-1, 9).

Pythagorean theorem

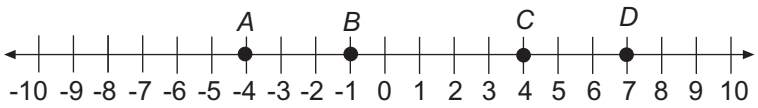
$$a^2 + b^2 = c^2$$

4. On the number line below, find the **distance** between A and B. Use either of the methods below.

   - **Use the number line and count in from both endpoints** of a line segment until you reach the middle to determine the midpoint.

   - **Use the Method One midpoint formula** and add the two endpoints together, then divide by two. **Show all your work.**

Method One midpoint formula
$\frac{a+b}{2}$

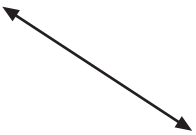


5. On the number line above, find the **midpoint** of $\overline{BD}$. Use either method above.

Use the list below to correctly describe each of the following lines. Write the answer on the line provided.

falling
horizontal
rising
vertical

_____ 6. 

_____ 7. 

_____ 8. 

_____ 9. 

Use the **slope formula** *below to find the* **slopes** *through each* **pair of points**.

slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

10. (3, -8), (5, 7)

11. (-2, 0), (6, -3)

Use the **slope-intercept form** *below to find the* **slope** *for each* **line**.

slope-intercept form

$$y = mx + b$$

12. $y = \frac{1}{2}x - 7$

13. $y = -2x + 6$

Use the **slope formula** *below to find the* **slopes** *of* $\overleftrightarrow{AB}$ *and* $\overleftrightarrow{CD}$*. Then* **multiply the slopes** *to determine if they are* **parallel**, **perpendicular**, *or* **neither**. **Show all your work.** *Write the answer on the line provided. Refer to page 373 number 1 for an example.*

slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

_____ 14. *A* (2, -5), *B* (4, 5), *C* (-3, 8), *D* (2, 7)

_____ 15. *A* (-4, 0), *B* (6, 1), *C* (4, 3), *D* (-6, 2)

Put equations in **slope-intercept form**. **Show all your work.** *Determine if the following* **lines** *are* **parallel**, **perpendicular**, *or* **neither**. *Write the answer on the line provided.*

_____ 16. $y = \frac{1}{2}x - 6$

$y = 2x + 6$

_____ 17. $y = -\frac{3}{2}x + 4$

$y = -\frac{3}{2}x + 6$

_____ 18. $y = \frac{5}{8}x + 1$

$y = -\frac{8}{5}x - 3$

Unit 7: Sizing Things Up

This unit will illustrate the difference between shape and size as they relate to the concepts of congruency and similarity.

Unit Focus

Measurement

- Relate the concepts of measurement to similarity and proportionality in real world applications. (MA.B.1.4.3)

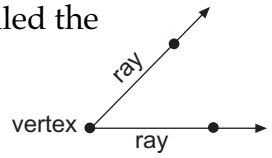
Geometry and Spatial Sense

- Understand geometric concepts such as perpendicularity, parallelism, tangency, congruency, similarity, reflections, symmetry, and transformations including flips, slides, turns, enlargements, rotations, and fractals. (MA.C.2.4.1)
- Represent and apply geometric properties and relationships to solve real-world and mathematical problems including ratio and proportion. (MA.C.3.4.1)

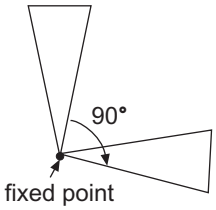
Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

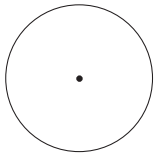
angle (∠) ..... two rays extending from a common endpoint called the vertex; measures of angles are described in degrees (°)



angle of rotation ..... the degrees an angle of a geometric figure is rotated or turned
Example: The amount of rotation is usually expressed in the number of degrees, such as a 90° rotation.



circle ..... the set of all points in a plane that are all the same distance from a given point called the center



congruent (≅) ..... figures or objects that are the same shape and size

corresponding ..... in the same location in their respective figures

corresponding angles and sides ..... the matching angles and sides in similar figures

cross multiplication ..... a method for solving and checking proportions; a method for finding a missing numerator or denominator in equivalent fractions or ratios by making the cross products equal

Example: To solve this proportion:

$$\frac{n}{9} = \frac{8}{12}$$

$$12 \times n = 9 \times 8$$
$$12n = 72$$
$$n = \frac{72}{12}$$
$$n = 6$$

Solution:

$$\frac{6}{9} = \frac{8}{12}$$

degree (°) ..... common unit used in measuring angles

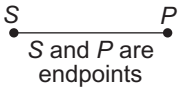
denominator ..... the bottom number of a fraction, indicating the number of equal parts a whole was divided into

Example: In the fraction $\frac{2}{3}$ the denominator is 3, meaning the whole was divided into 3 equal parts.

distributive property ..... the product of a number and the sum or difference of two numbers is equal to the sum or difference of the two products

Example: $x(a + b) = ax + bx$

endpoint ..... either of two points marking the end of a line segment

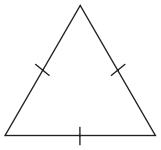


equation ..... a mathematical sentence in which two expressions are connected by an equality symbol
Example: $2x = 10$

equiangular ..... a figure with all angles congruent

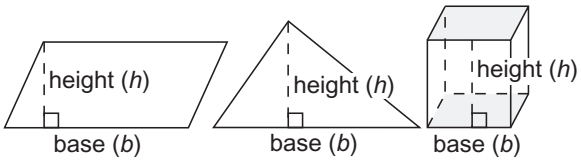
equilateral ..... a figure with all sides congruent

equilateral triangle ..... a triangle with three congruent sides



fraction ..... any part of a whole
Example: One-half written in fractional form is $\frac{1}{2}$.

height (*h*) ..... a line segment extending from the vertex or *apex* (highest point) of a figure to its base and forming a right angle with the base or plane that contains the base

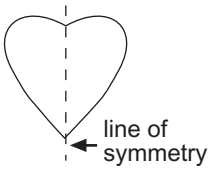


horizontal ..... parallel to or in the same plane of the horizon

integers ..... the numbers in the set
{… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

length (l) ..... a one-dimensional measure that is the measurable property of line segments

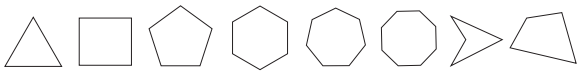
line of symmetry ..... a line that divides a figure into two congruent halves that are mirror images of each other



numerator ..... the top number of a fraction, indicating the number of equal parts being considered
Example: In the fraction $\frac{2}{3}$, the numerator is 2.

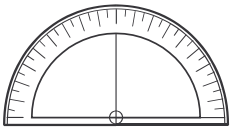
perimeter (P) ..... the distance around a polygon

polygon ..... a closed-plane figure, having at least three sides that are line segments and are connected at their endpoints
Example: triangle (3 sides), quadrilateral (4 sides), pentagon (5 sides), hexagon (6 sides), heptagon (7 sides), octagon (8 sides); concave, convex



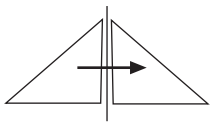
proportion ..... a mathematical sentence stating that two ratios are equal
Example: The ratio of 1 to 4 equals 25 to 100, that is $\frac{1}{4} = \frac{25}{100}$.

protractor ..... an instrument used for measuring and drawing angles



ratio ..... the comparison of two quantities
Example: The ratio of a and b is a:b or $\frac{a}{b}$, where $b \neq 0$.

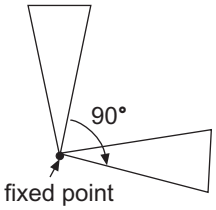
reflection ..... a transformation that produces the mirror image of a geometric figure over a line of reflection; also called a *flip*



reflectional symmetry ..... when a figure has at least one line which splits the image in half, such that each half is the mirror image or reflection of the other; also called *line symmetry* or *mirror symmetry*

regular polygon ..... a polygon that is both *equilateral* (all sides congruent) and *equiangular* (all angles congruent)

rotation ..... a transformation of a figure by turning it about a center point or axis; also called a *turn*
Example: The amount of rotation is usually expressed in the number of degrees, such as a 90° rotation.



rotational symmetry ..... when a figure can be turned less than 360 degrees about its center point to a position that appears the same as the original position; also called *turn symmetry*

rounded number ..... a number approximated to a specified place

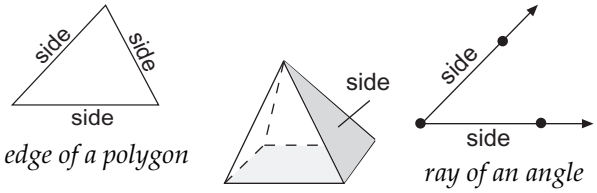
Example: A commonly used rule to round a number is as follows.

- If the digit in the first place after the specified place is 5 or more, *round up* by adding 1 to the digit in the specified place ($\underline{4}61$ rounded to the nearest hundred is 500).
- If the digit in the first place after the specified place is less than 5, *round down* by *not* changing the digit in the specified place ($\underline{4}41$ rounded to the nearest hundred is 400).

scale factor ..... the constant that is multiplied by the lengths of each side of a figure that produces an image that is the same shape as the original figure

side ..... the edge of a polygon, the face of a polyhedron, or one of the rays that make up an angle

Example: A triangle has three sides.

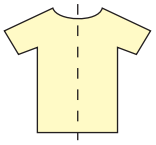


edge of a polygon *face of a polyhedron* *ray of an angle*

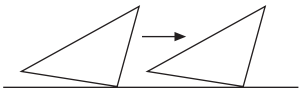
similar figures (~) ..... figures that are the same shape, have corresponding congruent angles, and have corresponding sides that are proportional in length

solve ..... to find all numbers that make an equation or inequality true

symmetry ..... a term describing the result of a line drawn through the center of a figure such that the two halves of the figure are reflections of each other across the line

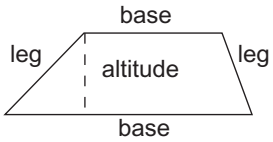


translation ..... a transformation in which every point in a figure is moved in the same direction and by the same distance; also called a *slide*

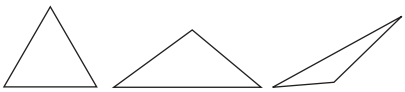


translational symmetry ..... when a figure can slide on a plane (or flat surface) without turning or flipping and with opposite sides staying congruent

trapezoid ..... a quadrilateral with just one pair of opposite sides parallel



triangle ..... a polygon with three sides



value (of a variable) ..... any of the numbers represented by the variable

variable ..... any symbol, usually a letter, which could represent a number

vertical ..... at right angles to the horizon; straight up and down

Unit 7: Sizing Things Up

Introduction

Students should be able to see that changing the size of a geometric figure can occur without changing the shape of a figure. Working with ratios and proportions will help the students understand the relationship between congruence and similarity.

Lesson One Purpose

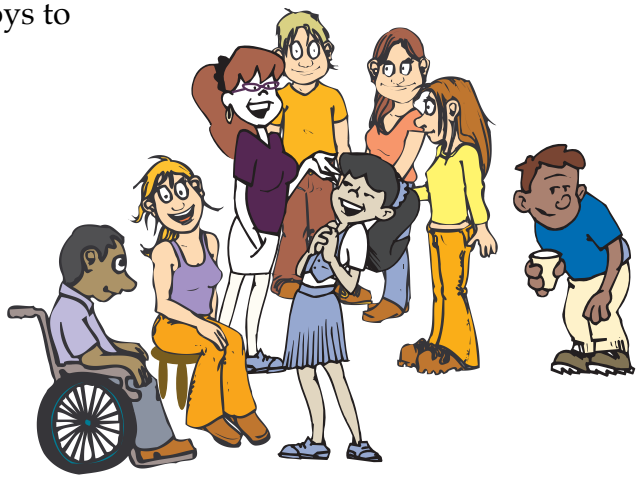
- Relate the concepts of measurement to similarity and proportionality in real world applications. (MA.B.1.4.3)
- Represent and apply geometric properties and relationships to solve real-world and mathematical problems including ratio and proportion. (MA.C.3.4.1)

Ratios and Proportions

Ratio is another word for a **fraction**. It is the comparison of two quantities: the **numerator** (top number of a *fraction*) and the **denominator** (bottom number of a fraction). For instance, if a classroom has 32 students and 20 of them are girls, we can say that the *ratio* of the number of girls to the number of students in the class is $\frac{20}{32} = \frac{5}{8}$ or 5:8. There are several other comparisons we can make using the information. We could compare the number of boys to the number of students, $\frac{12}{32} = \frac{3}{8}$.

What about the number of boys to the number of girls? $\frac{12}{20} = \frac{3}{5}$.

Or, the number of girls to the number of boys? $\frac{20}{12} = \frac{5}{3}$.



When two ratios are equal to each other, we have formed a **proportion**. A *proportion* is a mathematical sentence stating that two ratios are equal.

$$\frac{6}{9} = \frac{2}{3}$$

There are several properties of proportions that will be useful as we continue through this unit.

- We could *switch* the 6 with the 3 and still have a *true* proportion (Example 1).
- We could *switch* the 2 with the 9 and still have a *true* proportion (Example 2).
- We could even *flip* both fractions over and still have a *true* proportion (Example 3).

Example 1	**Example 2**	**Example 3**
$\frac{6}{9} = \frac{2}{3}$	$\frac{6}{9} = \frac{2}{3}$	$\frac{6}{9} = \frac{2}{3}$
$\frac{3}{9} = \frac{2}{6}$	$\frac{6}{2} = \frac{9}{3}$	$\frac{9}{6} = \frac{3}{2}$

Proportions are also very handy to use for problem solving. We use a process that involves **cross multiplying**, then **solve** the resulting **equation**. Look at the example below as we *solve* the *equation* and find the **value of the variable**.

$$\frac{3}{5} = \frac{x}{x+6}$$ ← cross multiply

$$3(x+6) = 5x$$
$$3x + 18 = 5x$$ ← distribute (**distributive property**)

$$3x - 3x + 18 = 5x - 3x$$
$$18 = 2x$$ ← subtract 3*x* from each side

$$\frac{18}{2} = \frac{2x}{2}$$
$$9 = x$$ ← divide each side by 2

Check your answer. Does $\frac{9}{9+6} = \frac{3}{5}$? Yes, $\frac{9}{15} = \frac{3}{5}$, so 9 is the correct *value* for x.

Try the following practice.

Practice

Find the **value of the variable** *in each of the following. Refer to previous pages as needed. Check your answers.* **Show all your work.**

1. $\frac{2}{x+1} = \frac{4}{x}$

2. $\frac{6}{z-2} = \frac{12}{4}$

3. $\frac{3}{2x-1} = \frac{7}{3x+1}$

4. $\frac{2}{x-9} = \frac{9}{x+12}$

5. $\frac{6}{x-1} = \frac{5}{x+2}$

6. $\frac{x-3}{18} = \frac{x+1}{30}$

7. $\frac{x-8}{x} = \frac{5}{7}$

8. $\frac{x+12}{2x+3} = \frac{5}{3}$

9. $\frac{2x}{x+3} = \frac{3}{2}$

Using Proportion Algebraically

We can use proportions in word problems as well. Here's an example.

In Coach Coffey's physical education class, the ratio of boys to girls is 3 to 4. If there are 12 boys in the class, how many girls are there?

When setting up proportions, you must have a plan and be consistent when you write the ratios. If you set up one ratio as $\frac{\text{boys}}{\text{girls}}$, then you must set up the other ratio in the same order, as $\frac{\text{boys}}{\text{girls}}$.

$\frac{3}{4} = \frac{12}{x}$ ← notice that both fractions indicate $\frac{\text{boys}}{\text{girls}}$

$3x = 4 \text{ x } 12$ ← cross multiply

$3x = 48$

$\frac{3x}{3} = \frac{48}{3}$ ← divide each side by 3

$x = 16$

Check your answer. Does $\frac{12}{16} = \frac{3}{4}$? Yes, so 16 is the correct answer.

Now it is your turn to practice on the following page.

Practice

Use **proportions** *to solve the following. Refer to the previous pages as needed. Check your answers.* **Show all your work.**

1. The ratio of two **integers** {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …} is 13:6. The smaller integer is 54. Find the *larger* integer.

   Answer: _____

2. The ratio of two integers is 7:11. The larger integer is 187. Find the *smaller* integer.

   Answer: _____

3. A shopkeeper makes $85 profit when he sells $500 worth of clothing. At the same rate of profit, what will he make on a $650 sale?

   Answer: $ _____

4. A baseball player made 43 hits in 150 times at bat. At the same rate, how many hits can he expect in 1,050 times at bat?

   Answer: _____

5. The cost of a 1,600-mile bus trip is $144. At the same rate per mile, what will be the cost of a 650-mile trip?

   Answer: $ _____

6. On a map, 19 inches represents 250 miles. What **length** on the map will represent 600 miles?

   Answer: _____ miles

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. the comparison of two quantities

_____ 2. the numbers in the set {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

_____ 3. a mathematical sentence in which two expressions are connected by an equality symbol

_____ 4. to find all numbers that make an equation or inequality true

_____ 5. the bottom number of a fraction, indicating the number of equal parts a whole was divided into

_____ 6. the top number of a fraction, indicating the number of equal parts being considered

_____ 7. a mathematical sentence stating that two ratios are equal

_____ 8. $x(a + b) = ax + bx$

_____ 9. any part of a whole

_____ 10. a one-dimensional measure that is the measurable property of line segments

_____ 11. a method for solving and checking proportions; a method for finding a missing numerator or denominator in equivalent fractions or ratios by making the cross products equal

A. cross multiplication

B. denominator

C. distributive property

D. equation

E. fraction

F. integers

G. length (l)

H. numerator

I. proportion

J. ratio

K. solve

Lesson Two Purpose

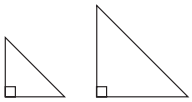
- Relate the concepts of measurement to similarity and proportionality in real world applications. (MA.B.1.4.3)
- Understand geometric concepts such as perpendicularity, parallelism, tangency, congruency, similarity, reflections, symmetry, and transformations including flips, slides, turns, enlargements, rotations, and fractals. (MA.C.2.4.1)
- Represent and apply geometric properties and relationships to solve real-world and mathematical problems including ratio and proportion. (MA.C.3.4.1)

Similarity and Congruence

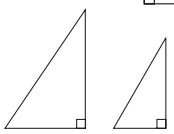
Geometric figures that are exactly the same shape, but not necessarily the same size, are called **similar figures (~)**. In *similar figures*, all the pairs of **corresponding angles** are the same measure, and all the pairs of **corresponding sides** are in the same ratio. This ratio, in its reduced form, is called the **scale factor**. When all pairs of *corresponding sides* are in the same ratio as the *scale factor*, we say that the **sides** are in proportion.

There are some geometric figures that are always similar.

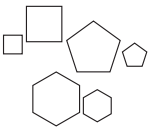
1. All **triangles** whose **angles (∠)** measures of **degree (°)** are 45°, 45°, and 90° are similar to each other.



2. All *triangles* whose *angles (∠)* measures of *degree* (°) are 30°, 60°, and 90° are similar to each other.



3. All **regular polygons** with the same number of *sides* are similar to each other.



Remember: A *regular polygon* is a **polygon** that is **equilateral** and **equiangular**. Therefore, all its sides are **congruent (≅)** and all angles are *congruent (≅)*.

Note: Circles seem to be similar, but since they have no angle measures, we don't include them in this group.

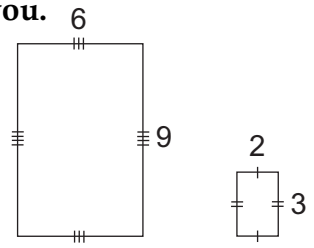
Practice

Look at each **pair of figures** *below. Determine if they are* **similar or not to each other**.

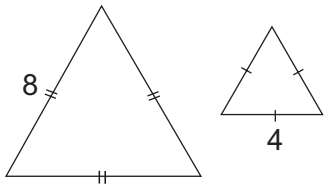
- *Write* **yes** *if they are similar.*
- *Write* **no** *if they are* not *similar.*
- *If they are similar, write the* **scale factor**.

The first one has been done for you.

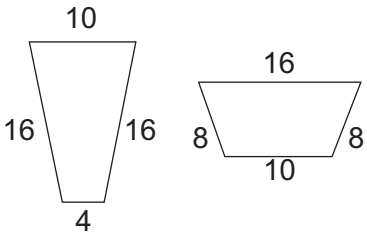
yes; 3:1 1.



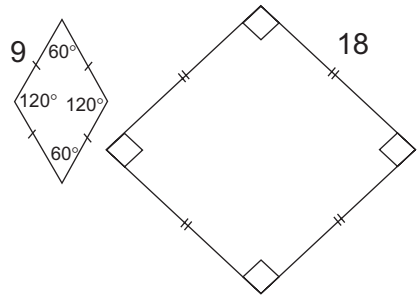
_____ 2.



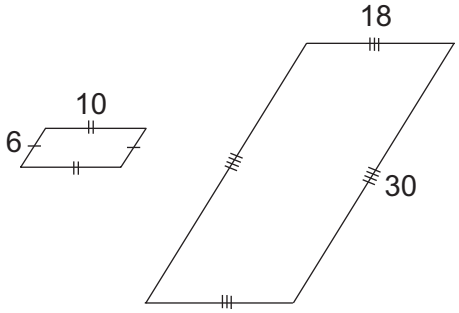
_____ 3.



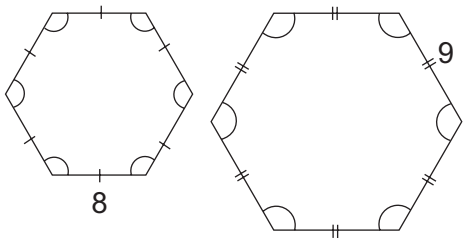
_____ 4.



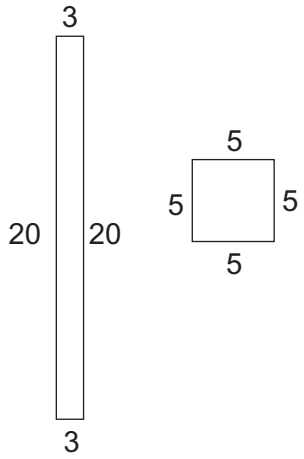
_____ 5.



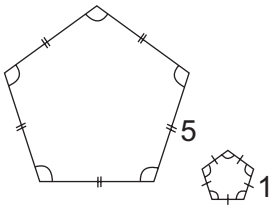
_____ 6.



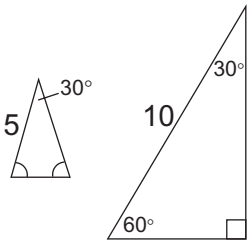
_____ 7.



_____ 8.



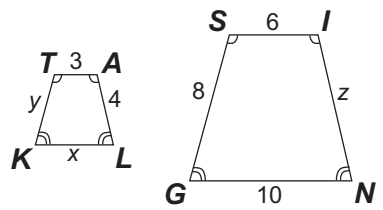
_____ 9.



Using Proportions Geometrically

If we know two shapes are similar, and we know some of the lengths, we often can find some of the other measures. Look at the two similar figures below. We have labeled the trapezoids *TALK* and *SING*.

Trapezoids *TALK* and *SING*



By locating the corresponding angles, we can say that

Trapezoid *TALK* ~ **Trapezoid** *SING*.

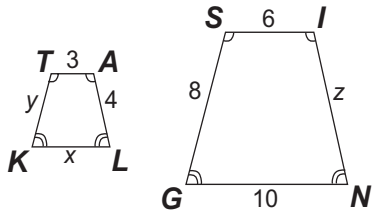
Note: ~ is the symbol for similar.

To find the values of x, y, and z, we must first find a pair of corresponding sides with lengths given.

- Side *TA* and side *SI* are a pair of corresponding sides.
- We are given that $TA = 3$ and $SI = 6$.
- So, we can set up a ratio $\frac{TA}{SI} = \frac{3}{6}$.
- When we reduce the ratio, we get the scale factor, which is $\frac{1}{2}$.
- This means that every length in *TALK* is one-half its **corresponding** length in *SING*.

Now we can use the scale factor to make proportions and find x, y, and z. Remember to be consistent as you set up the proportions. Since my scale factor was determined by a comparison of *TALK* to *SING*, I will continue in that order: ($\frac{TALK}{SING}$).

Trapezoids *TALK* and *SING*



$\frac{1}{2} = \frac{x}{10}$	$\frac{1}{2} = \frac{y}{8}$	$\frac{1}{2} = \frac{4}{z}$
$2x = 10$	$2y = 8$	$8 = 1z$
$x = 5$	$y = 4$	$8 = z$

What is the **perimeter**, or distance around the *polygon*, of *TALK*?

Did you get 16?

Can you guess the *perimeter* of *SING*?

If you guessed 32, you are correct.

Does it make sense that the perimeters should be in the same ratio as the scale factor?

Yes, because the perimeters are corresponding lengths. In addition, all *corresponding* lengths in similar figures are in proportion!

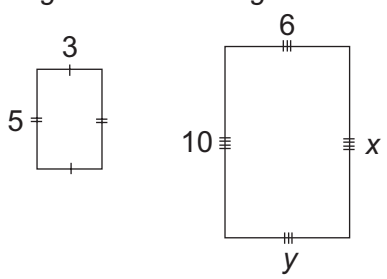
Practice

Find the following for each pair of **similar figures** *below.*

- *scale factor (SF)*
- $x =$
- $y =$
- $P_1 =$ *perimeter of figure 1*
- $P_2 =$ *perimeter of figure 2*

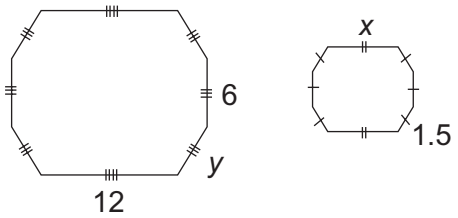
Refer to the previous pages as needed. **The first one has been done for you.**

1. *figure 1* *figure 2*



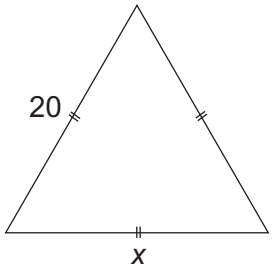
SF 1:2 ; $x =$ 10 ; $y =$ 6 ; P_1 16

2. *figure 1* *figure 2*



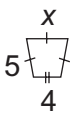
SF _____ ; $x =$ _____ ; $y =$ _____ ; P_1 _____

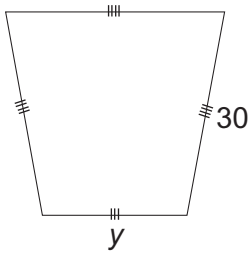
3. *figure 1* *figure 2*



SF _____ ; $x =$ _____ ; $y =$ _____ ; P_2 _____

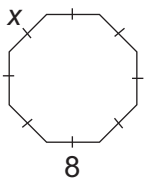
4. *figure 1* *figure 2*

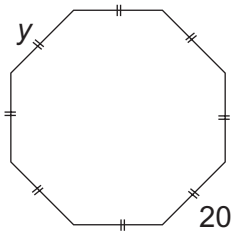




SF _____ ; $x =$ _____ ; $y =$ _____ ; P_1 _____

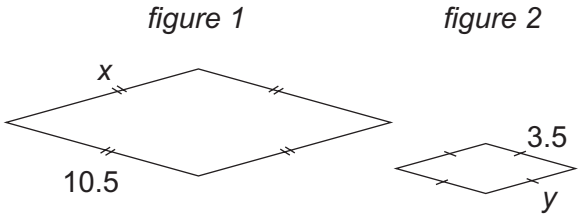
5. *figure 1* *figure 2*





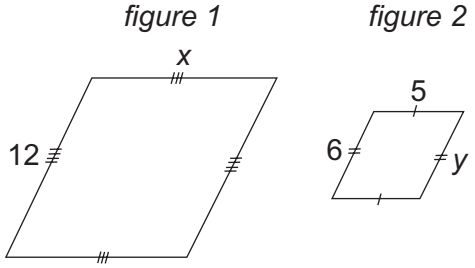
SF _____ ; $x =$ _____ ; $y =$ _____ ; P_1 _____

6.



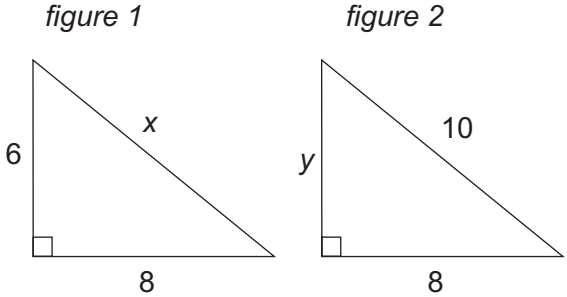
SF _____ ; $x =$ _____ ; $y =$ _____ ; P_2 _____

7.



SF _____ ; $x =$ _____ ; $y =$ _____ ; P_2 _____

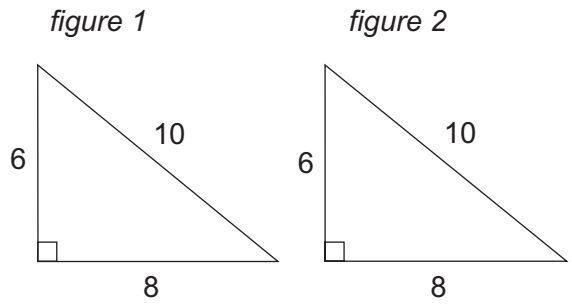
8.



SF _____ ; $x =$ _____ ; $y =$ _____ ; P_1 _____ ; P_2 _____

Using Proportions to Find Heights

Look at the figures below. They are from number 8 in the previous practice.

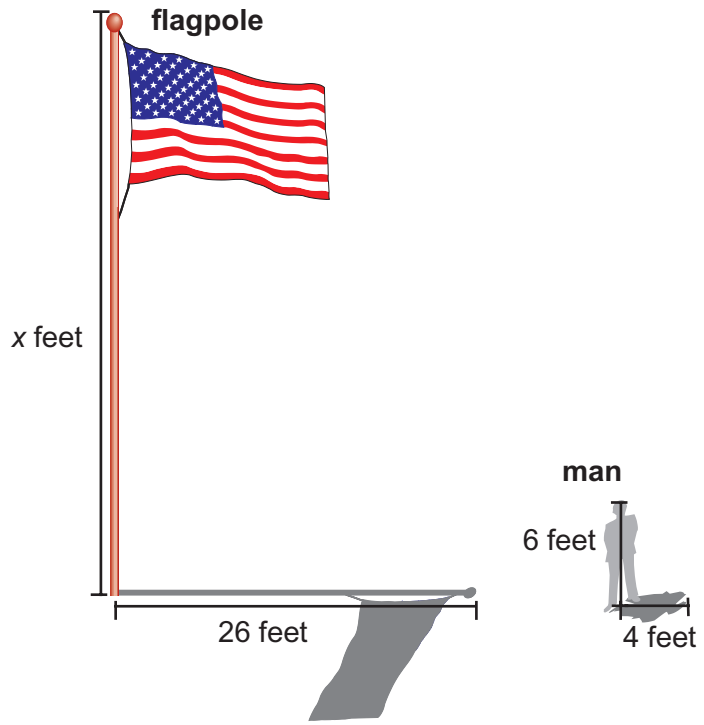


Here is what we know about *figure 1* and *figure 2* above.

- Their scale factor is $\frac{1}{1}$. This makes all the pairs of corresponding sides the same length.
- We already knew that their corresponding angles were the same measure because we knew that they were similar. This makes the triangles identical to each other.

Geometric figures that are *exactly* the same *shape* and *exactly* the same *size* are *congruent* to each other. The symbol for congruence, $\cong$, is a lot like the symbol for similar, but the equal sign, =, underneath it tells us that two things are *exactly* the same *size*.

We can use proportions to find the lengths of some items that would be difficult to measure. For instance, if we needed to know the height of a flagpole without having to inch our way up, we could use proportions. See the example on the following page.



A 6-foot man casts a 4-foot shadow at the same time a flagpole casts a 26-foot shadow. Find the **height (*h*)** of the flagpole.

To solve a problem like this, set up a proportion comparing corresponding parts.

$$\frac{\text{man's height}}{\text{man's shadow}} = \frac{\text{flagpole's height}}{\text{flagpole's shadow}}$$

$$\frac{6}{4} = \frac{x}{26}$$

cross multiply

$$4x = 6 \text{ x } 26$$

$$4x = 156$$

divide both sides by 4

$$\frac{4x}{4} = \frac{156}{4}$$

$$x = 39 \text{ feet}$$

Now try the following practice.

Practice

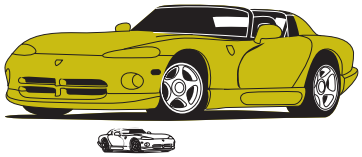
Use **proportions** *to solve the following. Refer to the previous pages as needed.*
Round to the nearest tenth. Show all your work.

1. A tree casts a 50-foot shadow at the same time a 4-foot fence post casts a 3-foot shadow. How tall is the tree?

   Answer: _____ feet

2. If the scale factor for a miniature toy car and a real car is 1 to 32 and the windshield on the toy car is 2 inches long, how long is the windshield on the real car?

   Answer: _____ inches



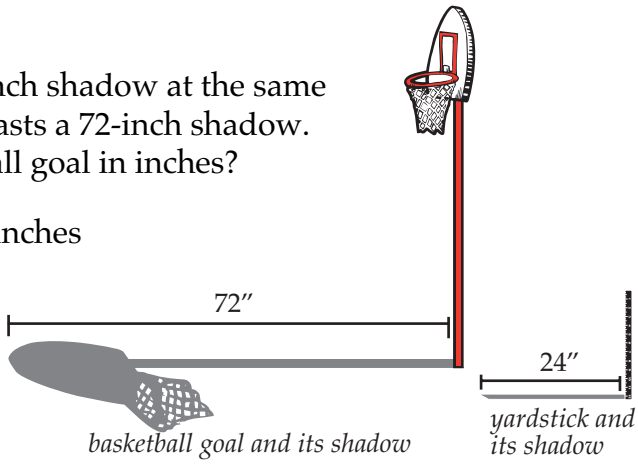
scale factor = $\frac{1}{32}$

3. The goal post on the football field casts an 18-foot shadow. The 4-foot water cooler casts a 5-foot shadow. How tall is the goal post?

   Answer: _____ feet

4. A yardstick casts a 24-inch shadow at the same time a basketball goal casts a 72-inch shadow. How tall is the basketball goal in inches?

   Answer: _____ inches



5. A photo that is 4 inches by 6 inches needs to be enlarged so that the shorter sides are 6 inches. What will be the length of the enlargement?

   Answer: _____ inches

Lesson Three Purpose

- Relate the concepts of measurement to similarity and proportionality in real world applications. (MA.B.1.4.3)
- Understand geometric concepts such as perpendicularity, parallelism, tangency, congruency, similarity, reflections, symmetry, and transformations including flips, slides, turns, enlargements, rotations, and fractals. (MA.C.2.4.1)
- Represent and apply geometric properties and relationships to solve real-world and mathematical problems including ratio and proportion. (MA.C.3.4.1)

Symmetry and Transformation

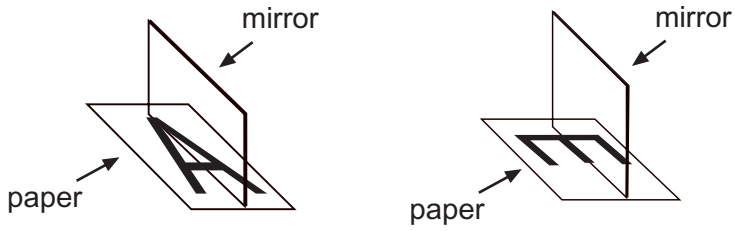
Reflectional Symmetry

A figure has **reflectional symmetry** if there is at least one line which splits the image in half. Once split, one side is the mirror image or **reflection** of the other.

If a line is drawn through the center of a figure and the two halves are congruent, the figure has reflectional symmetry. This is often called *line* **symmetry** or *mirror symmetry*. And the line that divides a figure into two congruent halves that are mirror images of each other is called a **line of symmetry**.

Remember: It is possible to have more than one line of reflectional symmetry.

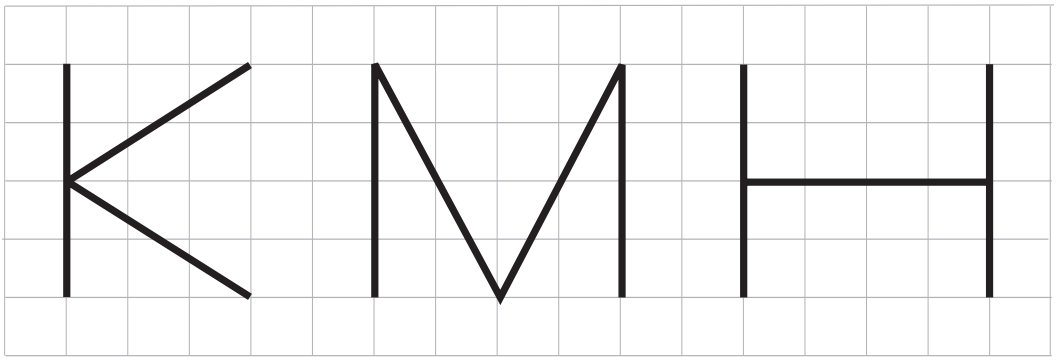
Try this with the capital letters A and E.



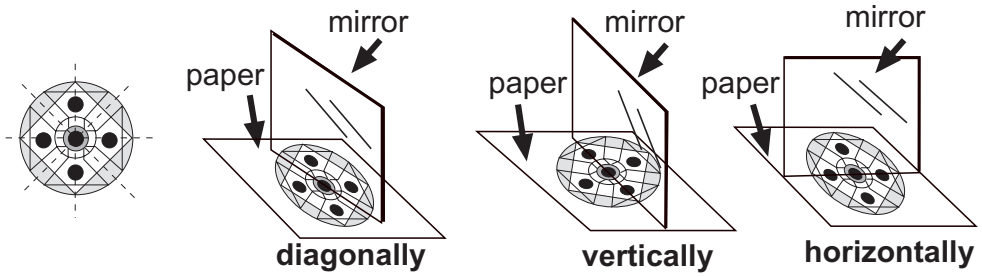
When testing figures for reflectional symmetry, look for a line of symmetry.

- You might fold the figure on a line to test for a match.
- You might use a mirror or reflecting device on a line to test for a match.
- You might use tracing paper to trace half of the figure and *flip* the tracing over a line to test the image for a match.

The following letters illustrate reflectional symmetry, either **horizontally (↔)** or **vertically (↕)** or both.

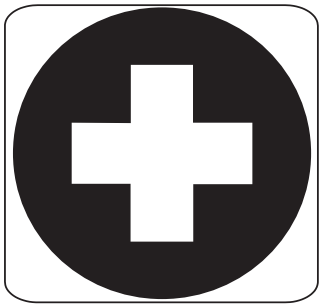


The following design also illustrates reflectional symmetry *horizontally*, *vertically*, and *diagonally* (⤢).



Practice

Determine if the **shapes** *of the following* **signs** *and their* **symbols** *illustrate* **reflectional symmetry**. *Below each sign write* **yes** *or* **no**.



1. _____ 2. _____ 3. _____

4. _____ 5. _____ 6. _____

Rotational Symmetry

If a figure can be **rotated** by *turning* it less than 360 degrees about its center point to a position that appears the same as the original position, then the figure has **rotational symmetry**. This is often called *turn symmetry*. Try this with an **equilateral triangle**.

- If you cut an identical triangle from a piece of paper,

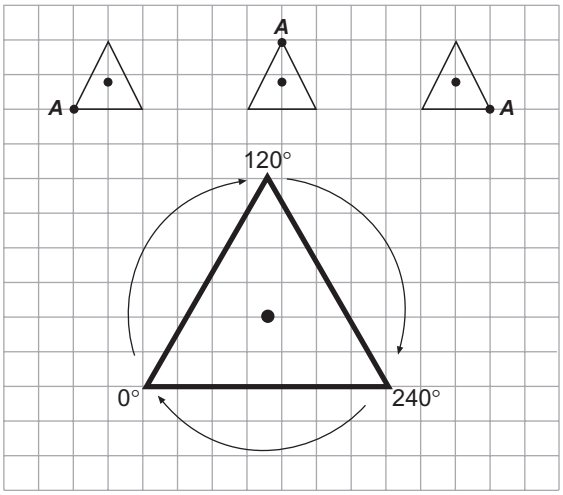
  — lay it on top of the original triangle, and

  — turn the figure about its center point 120°,

  — the turned figure appears the same as the original.

- If you turn it another 120° for a total of 240°, the result is the same.

- A third turn of 120° returns the triangle to its original position. The **angle of rotation** for this figure is 120°.



Finding the center point is sometimes challenging, but keep trying. Once you find the center and *rotate* successfully to find a match, the angle must be measured to determine the *angle of rotation*.

When testing figures for rotational symmetry, look for center point symmetry, and then do the following.

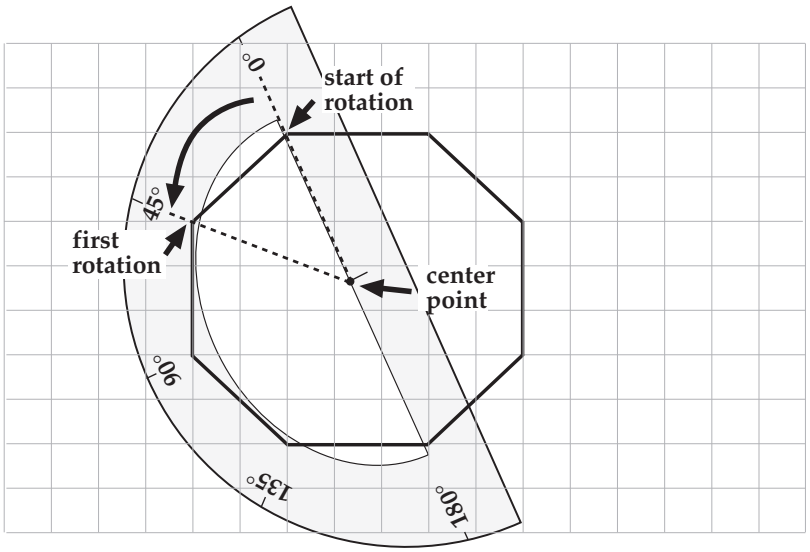
1. trace the figure using tracing paper
2. rotate the tracing to test for a match

The following figures illustrate rotational symmetry. Each figure can be rotated around a center point. The image produced by the rotation is congruent to the original. Use a **protractor** to measure the angle of rotation.

- Place the center of the *protractor* on the center point of the figure.
- Line up the 0° mark with the start of the rotation.
- Use a straightedge to extend the line for easier reading of the measure.
- Rotate the figure and mark the place the rotated figure (the image) matches the original figure.
- Extend the line with a straightedge and note the degree on the protractor.

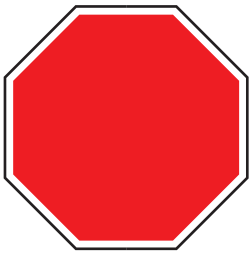
The degree the rotation stopped at is called the angle of rotation.

Note that the measure of the angle of rotation for the figure below is 45°.

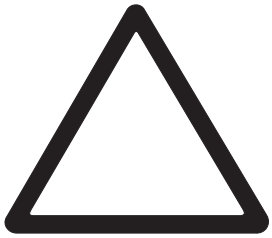


Practice

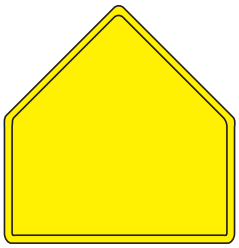
Determine if the **shapes** *of the following* **signs** *illustrate* **rotational symmetry**. *Below each sign write* **yes** *or* **no**. *If* **yes**, *write the measure of the* **angle of rotation**.



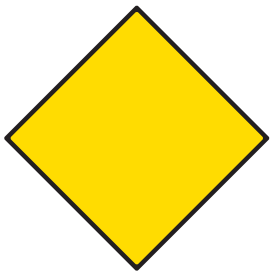
1. _____



2. _____

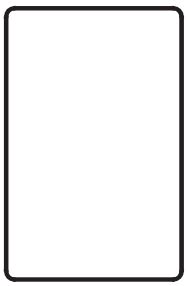


3. _____



4. _____

5. _____



6. _____

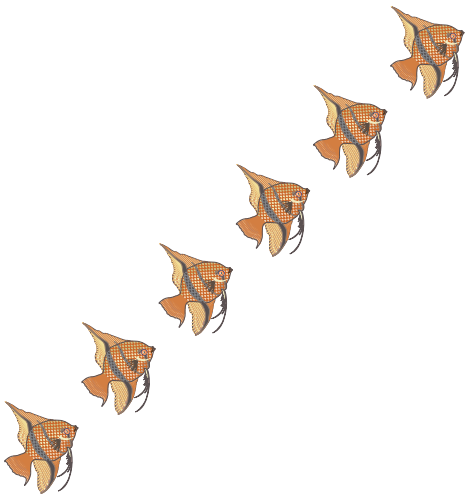
Practice

Draw **three figures** *that illustrate* **rotational symmetry**. *Write the* **angle of rotation** *below each of your illustrations.*

Translational Symmetry

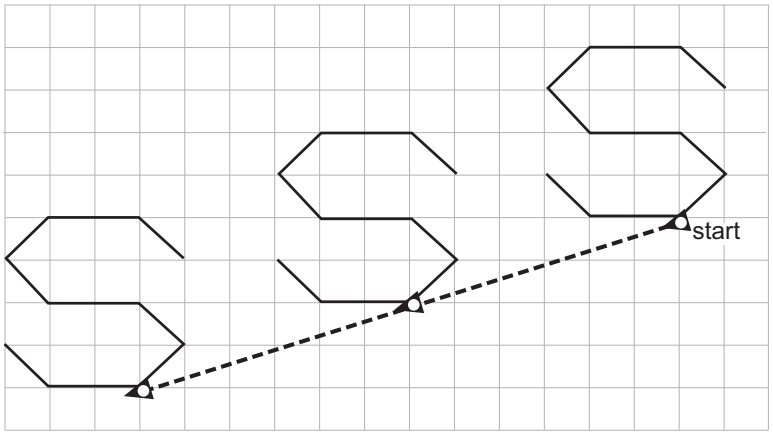
When one slides a figure a specific distance in a straight line from one place to another, **translational symmetry** occurs. The distance of the slide, or **translation**, and the direction of the slide are the important elements here. They are often illustrated by an arrow with its **endpoint** on one point of the first figure and an arrow on the corresponding point in the next figure (see grid below).

(The figure slides one space to the right.)



(The figure slides one space to the right and one up.)

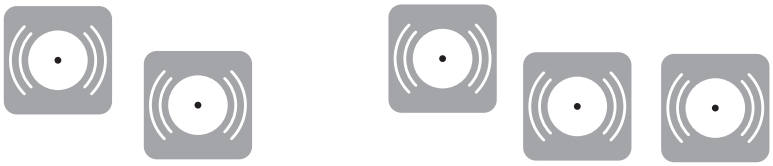
The figure below illustrates *translational symmetry*.



Practice

Determine if the following **signs** *illustrate* **translational symmetry**. *Below each grouping of signs write* **yes** *or* **no**.

1. _____



2. _____

3. _____

Practice

Draw **three figures** *that illustrate* **translational symmetry**. *Provide an arrow to specify the* **length** *and* **direction** *of your* **slide** *for each of your illustrations, like the bottom figure on page 437.*

- You might use tracing paper to trace the figure and slide it in a straight line along its *endpoint* to test for a match.
- As you test figures for translational symmetry, an endpoint is sought to slide the figure along a straight line.

Practice

Use the list below to complete the following statements.

congruent (≅)	**perimeter (*P*)**	**regular polygon**
equiangular	**proportion**	**scale factor**
equilateral	**ratio**	

1. A figure with all angles congruent is called

   _____ .

2. The comparison of two quantities is a _____ .

3. Figures or objects that are the same shape and size are said to be

   _____ .

4. _____ is the distance around a polygon.

5. A figure with all sides congruent is called

   _____ .

6. _____ is a mathematical sentence stating that two ratios are equal.

7. The constant that is multiplied by the lengths of each side of a figure that produces an image that is the same shape as the original figure is the _____ .

8. A polygon that is both equilateral and equiangular is called a

   _____ .

Practice

Use the list below to complete the following statements.

angle of rotation	**rotation** or **turn**
line of symmetry	**translation** or **slide**
reflection or **flip**	

1. A _____ is a line that divides a figure into two congruent halves that are mirror images of each other.

2. A _____ is a transformation that produces the mirror image of a geometric figure.

3. The degrees an angle of a geometric figure is rotated or turned is called the _____ .

4. A transformation in which every point in a figure is moved in the same direction and by the same distance is called a _____ .

5. A _____ is a transformation of a figure by turning it about a center point or axis.

Unit Review

Find the **value of the variable** *in the following. Check your answers.* **Show all your work.**

1. $\frac{x+2}{x} = \frac{5}{3}$

2. $\frac{4}{3x+1} = \frac{7}{5x-2}$

3. $\frac{3x}{x+7} = \frac{2}{3}$

4. $\frac{9x-1}{7} = \frac{3x-11}{2}$

Use **proportions** *to solve the following. Check your answers.* **Show all your work.**

5. The ratio of two integers is 9:7. The smaller integer is 448. Find the *larger* integer.

   Answer: _____

6. The ratio of two integers is 6:11. The larger integer is 88. Find the *smaller* integer.

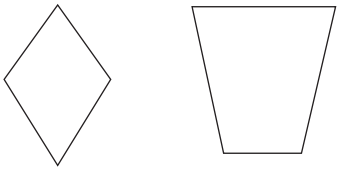
   Answer: _____

7. The cost of 24 pounds of rice is $35. At the same rate, what would 5 pounds of rice cost? **Round to the nearest whole cent.**

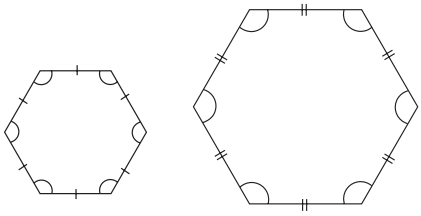
   Answer: $ _____

Look at each **pair of figures** *below. Determine if they are* **similar to each other**. *Write* **yes** *if they are similar. Write* **no** *if they are* not *similar.*

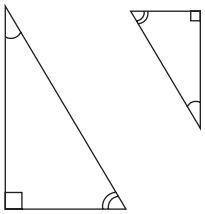
_____ 8.



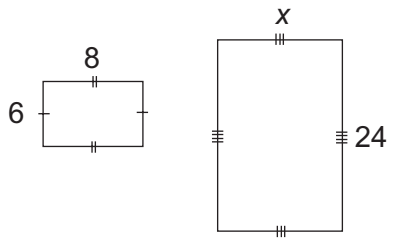
_____ 9.



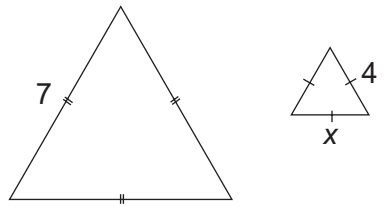
_____ 10.



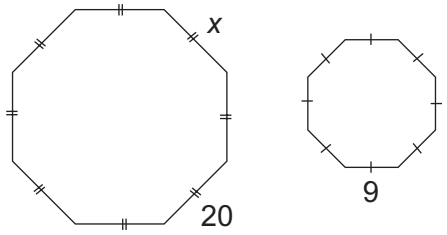
Each **pair of figures** *below is* **similar**. *Find the* **scale factor** *and* **value of the variable**.

11. 

SF = _____ ; x = _____

12. 

SF = _____ ; x = _____

13. 

SF = _____ ; x = _____

Use **proportions** *to solve the following.* **Show all your work.**

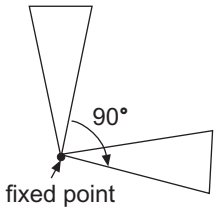
14. A tree casts a 40-foot shadow at the same time a 6-foot post casts an 8-foot shadow. How tall is the tree?

    Answer: _____ feet

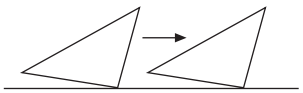
15. A 3.5-foot-tall mailbox casts a shadow of 5 feet at the same time a light pole casts a 20-foot shadow. How tall is the light pole?

    Answer: _____ feet

Circle the letter that represents the **type of transformation** *each* **figure shows**.

16. 

    a. reflection or flip
    b. rotation or turn
    c. translation or slide

17. 

    a. reflection or flip
    b. rotation or turn
    c. translation or slide

Unit 8: (X, Y) Marks the Spot!

This unit allows students to solve equations algebraically and graphically.

Unit Focus

Algebraic Thinking

- Use systems of equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

area (*A*) ..... the measure, in square units, of the inside region of a two-dimensional figure
Example: A rectangle with sides of 4 units by 6 units contains 24 square units or has an area of 24 square units.

axes (of a graph) ..... the horizontal and vertical number lines used in a coordinate plane system; (singular: *axis*)

coefficient ..... the number part in front of an algebraic term signifying multiplication
Example: In the expression $8x^2 + 3xy - x$,

- the coefficient of x^2 is 8 (because $8x^2$ means $8 \bullet x^2$)
- the coefficient of xy is 3 (because $3xy$ means $3 \bullet xy$)
- the coefficient of $-x$ is 1 (because $-1 \bullet x = -x$).

In general algebraic expressions, coefficients are represented by letters that may stand for numbers. In the expression $ax^2 + bx + c = 0$, a, b, and c are coefficients, which can take any number.

consecutive ..... in order
Example: 6, 7, 8 are consecutive whole numbers and 4, 6, 8 are consecutive even numbers.

coordinate grid or plane ..... a two-dimensional network of horizontal and vertical lines that are parallel and evenly-spaced; especially designed for locating points, displaying data, or drawing maps

distributive property ..... the product of a number and the sum or difference of two numbers is equal to the sum or difference of the two products
Example: $x(a + b) = ax + bx$

equation ..... a mathematical sentence in which two expressions are connected by an equality symbol
Example: $2x = 10$

equivalent expressions ..... expressions that have the same value but are presented in a different format using the properties of numbers

even integers ..... any integer divisible by 2
Example: {-4, -2, 0, 2, 4}

factor ..... a number or expression that divides evenly into another number; one of the numbers multiplied to get a product
Example: 1, 2, 4, 5, 10, and 20 are factors of 20 and $(x + 1)$ is one of the factors of $(x^2 - 1)$.

factored form ..... a monomial expressed as the product of prime numbers and variables, where no variable has an exponent greater than 1

factoring ..... expressing a polynomial expression as the product of monomials and polynomials
Example: $x^2 - 5x + 4 = 0$
$(x - 4)(x - 1) = 0$

formula ..... a way of expressing a relationship using variables or symbols that represent numbers

graph ..... a drawing used to represent data
Example: bar graphs, double bar graphs, circle graphs, and line graphs

graph of an equation ..... all points whose coordinates are solutions of an equation

inequality ..... a sentence that states one expression is greater than (>), greater than or equal to (≥), less than (<), less than or equal to (≤), or not equal to (≠) another expression
Example: $a \neq 5$ or $x < 7$ or $2y + 3 \geq 11$

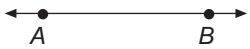
infinite ..... having no boundaries or limits

integers ..... the numbers in the set
{… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

intersect ..... to meet or cross at one point

length (*l*) ..... a one-dimensional measure that is the measurable property of line segments

line ($\overleftrightarrow{}$) ..... a collection of an infinite number of points in a straight pathway with unlimited length and having no width



monomial ..... a number, variable, or the product of a number and one or more variables; a polynomial with only *one* term

Examples: 8 x $4c$ $2y^2$ -3 $\frac{xyz^2}{9}$

negative integers ..... integers less than zero

negative numbers ..... numbers less than zero

odd integers ..... any integer *not* divisible by 2
Example: {-5, -3, -1, 1, 3, 5}

ordered pair ..... the location of a single point on a rectangular coordinate system where the first and second values represent the position relative to the x-axis and y-axis, respectively
Example: (x, y) or (3, -4)

parallel (||) ..... being an equal distance at every point so as to never intersect

point ..... a specific location in space that has no discernable length or width

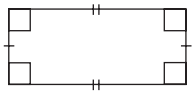
polynomial ..... a monomial or sum of monomials; any rational expression with no variable in the denominator
Examples: $x^3 + 4x^2 - x + 8$ $5mp^2$
$-7x^2y^2 + 2x^2 + 3$

positive integers ..... integers greater than zero

product ..... the result of multiplying numbers together
Example: In 6 x 8 = 48,
48 is the product.

quadratic equation ..... an equation in the form of
$ax^2 + bx + c = 0$

rectangle ..... a parallelogram with four right angles



simplify an expression ..... to perform as many of the indicated operations as possible

solution ..... any value for a variable that makes an equation or inequality a true statement
Example: In $y = 8 + 9$
$y = 17$ 17 is the solution.

solution set ({ }) ..... the set of values that make an equation or inequality true
Example: {5, -5} is the solution set for
$3x^2 = 75$.

solve ..... to find all numbers that make an equation or inequality true

standard form
(of a quadratic equation) ..... $ax^2 + bx + c = 0$, where a, b, and c are integers (not multiples of each other) and $a > 0$

substitute ..... to replace a variable with a numeral
Example: $8(a) + 3$
$8(5) + 3$

substitution ..... a method used to solve a system of equations in which variables are replaced with known values or algebraic expressions

sum ..... the result of adding numbers together
Example: In $6 + 8 = 14$,
14 is the sum.

system of equations ..... a group of two or more equations that are related to the same situation and share variables
Example: The solution to a system of equations is an ordered number set that makes all of the equations true.

table (or chart) ..... a data display that organizes information about a topic into categories

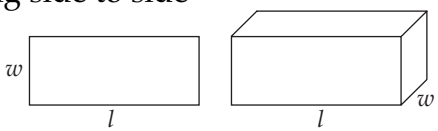
term ..... a number, variable, product, or quotient in an expression
Example: In the expression $4x^2 + 3x + x$, $4x^2$, $3x$, and x are terms.

value (of a variable) ..... any of the numbers represented by the variable

variable ..... any symbol, usually a letter, which could represent a number

vertical ..... at right angles to the horizon; straight up and down

width (w) ..... a one-dimensional measure of something side to side



Unit 8: (X, Y) Marks the Spot!

Introduction

Students should be able to blend their knowledge of equations, problem solving, and factoring to solve real-world problems.

Lesson One Purpose

- Use systems of equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

Quadratic Equations

When we **solve** an **equation** like $x + 7 = 12$, we remember that we must subtract 7 from both sides of the equal sign.

$$\begin{aligned} x + 7 &= 12 \\ x + 7 - 7 &= 12 - 7 \quad \longleftarrow \text{subtract 7 from both sides} \\ x &= 5 \end{aligned}$$

That leaves us with $x = 5$. We know that 5 is the only **solution** or **value** that can replace x and make the $x + 7 = 12$ true.

$$\begin{aligned} \text{If } x + 7 &= 12, \quad \text{and} \\ x &= 5 \quad \text{is true, then} \\ 5 + 7 &= 12. \end{aligned}$$

Suppose you have an *equation* that looks like $(x + 7)(x - 3) = 0$. This means there are two numbers, one in each set of parentheses, that when multiplied together, their **product** is 0. What kinds of numbers can be multiplied and equal 0?

Look at the following options.

$2 \times -2 = -4$ $\qquad$ $\frac{1}{5} \times 5 = 1$ $\qquad$ $-\frac{4}{7} \times \frac{7}{4} = -1$

The only way for numbers to be multiplied together with a result of zero is if one of the numbers is a 0.

$a \times 0 = 0$

Looking back at $(x + 7)(x - 3) = 0$, we understand that here are two **factors**, $(x + 7)$ and $(x - 3)$. The only way to multiply them and get a *product* of 0 is if one of them is equal to zero.

This leads us to a way to *solve* the equation. Since we don't know which of the terms equals 0, we cover all the options and assume either could be equal to zero.

If $x + 7 = 0$, then $x = -7$.

If $x - 3 = 0$, then $x = 3$.

We now have two options which could replace x in the original equation and make it true. Let's replace x with -7 and 3, one at a time.

$$\begin{aligned} (x + 7)(x - 3) &= 0 \\ (-7 + 7)(-7 - 3) &= 0 \\ (0)(-10) &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} (x + 7)(x - 3) &= 0 \\ (3 + 7)(3 - 3) &= 0 \\ (10)(0) &= 0 \\ 0 &= 0 \end{aligned}$$

Therefore, because either *value* of x gives us a true statement, we see that the **solution set** for $(x + 7)(x - 3) = 0$ is $\{-7, 3\}$.

Now you try the items in the following practice.

Practice

Find the **solution sets**. *Refer to pages 455 and 456 as needed.*

1. $(x + 4)(x - 2) = 0$ { _____ , _____ }

2. $(x - 5)(x + 3) = 0$ { _____ , _____ }

3. $(x - 5)(x - 7) = 0$ { _____ , _____ }

4. $(x + 6)(x + 1) = 0$ { _____ , _____ }

5. $(x - 2)(x - 2) = 0$ { _____ , _____ }

6. $(x + 18)(x - 23) = 0$ { _____ , _____ }

7. $x(x - 16) = 0$ { _____ , _____ }

8. $(x - 5)(2x + 6) = 0$ { _____ , _____ }

9. $(3x - 5)(5x + 10) = 0$ { _____ , _____ }

10. $(10x - 4)(x + 5) = 0$ { _____ , _____ }

Factoring to Solve Equations

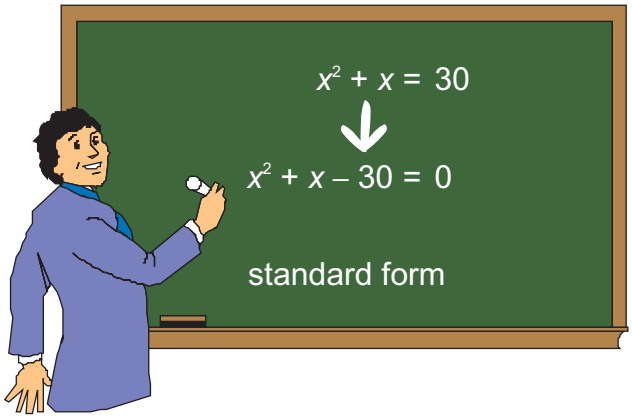
Often, equations are not given to us in **factored form** like those on the previous pages. Looking at $x^2 + x = 30$, we notice the x^2 **term** which tells us this is a **quadratic equation** (an equation in the form $ax^2 + bx + c = 0$). This term also tells us to be on the lookout for two answers in our *solution set*.

You may solve this problem by trial and error. However, we can also solve $x^2 + x = 30$ using a format called **standard form (of a quadratic equation).** This format is written with the *terms* in a special order:

- the x^2 term first
- then the x-term
- then the numerical term followed by $= 0$.

For our original equation,

$$\begin{aligned} x^2 + x &= 30 && \leftarrow \text{put in } \textit{standard form} \\ x^2 + x - 30 &= 30 - 30 && \leftarrow \text{subtract 30 from both} \\ x^2 + x - 30 &= 0 && \leftarrow \text{sides} \end{aligned}$$



Now that we have the proper format, we can factor the quadratic **polynomial**. (See Unit 2, Lesson 7 for a review).

Remember: Factoring expresses a *polynomial* as the product of **monomials** and polynomials.

Example 1

$$x^2 + x - 30 = 0$$
$$(x + 6)(x - 5) = 0$$

If $x + 6 = 0$, then $x = -6$.

If $x - 5 = 0$, then $x = 5$.

Therefore, the solution set is {-6, 5}.

Example 2

$$x^2 = 5x - 4$$
$$x^2 - 5x + 4 = 0 \quad \leftarrow \text{put in standard form}$$
$$(x - 4)(x - 1) = 0 \quad \leftarrow \text{factor}$$

If $x - 4 = 0$, then $\leftarrow$ set each factor = 0

$x = 4$.

If $x - 1 = 0$, then

$x = 1$.

$\{1, 4\}$ $\leftarrow$ write the solution set

Now it's your turn to practice on the following page.

Practice

Find the **solution sets**. *Refer to pages 459 and 460 as needed.*

1. $x^2 - x = 42$ { _____ , _____ }

2. $x^2 - 5x = 14$ { _____ , _____ }

3. $x^2 = -5x - 6$ { _____ , _____ }

4. $x^2 - x = 12$ { _____ , _____ }

5. $x^2 = 2x + 8$ { _____ , _____ }

6. $x^2 - 2x = 15$ { _____ , _____ }

7. $x^2 + 8x = -15$ { _____ , _____ }

8. $x^2 - 3x = 0$ { _____ , _____ }

9. $x^2 = -5x$ { _____ , _____ }

10. $x^2 - 4 = 0$ { _____ , _____ }

11. $x^2 = 9$ { _____ , _____ }

12. $3x^2 - 3 = 0$ { _____ , _____ }

13. $2x^2 = 18$ { _____ , _____ }

Practice

Use the list below to write the correct term for each definition on the line provided.

equation	**solution**
factor	**solve**
product	**value (of a variable)**

_____ 1. a mathematical sentence in which two expressions are connected by an equality symbol

_____ 2. to find all numbers that make an equation or inequality true

_____ 3. any of the numbers represented by the variable

_____ 4. any value for a variable that makes an equation or inequality a true statement

_____ 5. the result of multiplying numbers together

_____ 6. a number or expression that divides evenly into another number; one of the numbers multiplied to get a product

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____	1. an equation in the form of $ax^2 + bx + c = 0$	A. factored form
_____	2. a monomial or sum of monomials; any rational expression with no variable in the denominator	B. factoring
_____	3. the set of values that make an equation or inequality true	C. monomial
_____	4. expressing a polynomial expression as the product of monomials and polynomials	D. polynomial
_____	5. a number, variable, or the product of a number and one or more variables; a polynomial with only *one* term	E. quadratic equation
_____	6. a monomial expressed as the product of prime numbers and variables, where no variable has an exponent greater than 1	F. solution set ({ })

Solving Word Problems

We can also use the processes on the previous pages to solve word problems. Let's see how.

Example 1

Two **consecutive** (in order) **positive integers** (integers greater than zero) have a product of 110. Find the integers.

$$\begin{aligned} \text{let the 1st integer} &= x \\ \text{and the 2nd integer} &= x + 1 \end{aligned}$$

$$\begin{aligned} x(x+1) &= 110 \\ x^2 + x &= 110 \\ x^2 + x - 110 &= 0 \\ (x-10)(x+11) &= 0 \\ x - 10 &= 0 \quad \text{or} \quad x + 11 = 0 \\ x &= 10 \quad \text{or} \quad x = -11 \end{aligned}$$

Since the problem asked for *positive integers*, we must eliminate -11 as an answer. Therefore, the two integers are $x = 10$ and $x + 1 = 11$.

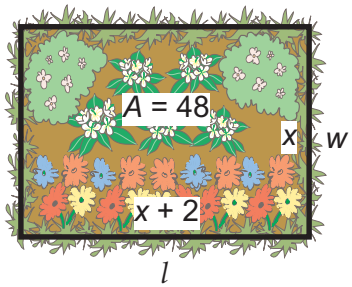
Remember: Integers are the numbers in the set {… , -4, -3, -2, -1, 9, 1, 2, 3, 4, …}.

Example 2

Billy has a garden that is 2 feet longer than it is wide. If the **area (*A*)** of his garden is 48 square feet, what are the dimensions of his garden?

If we knew the **width (*w*)**, we could find the **length (*l*)**, which is 2 feet longer. Since we don't know the *width*, let's represent it with x. The *length* will then be $x + 2$.

$$\begin{aligned} \text{width} &= x \\ \text{length} &= x + 2 \end{aligned}$$



The *area* (A) of a **rectangle** can be found using the **formula** length (l) times width (w).

$$\begin{aligned} A &= lw \\ A &= 48 \end{aligned}$$

$$\begin{aligned} \text{So,} \quad x(x+2) &= 48 \\ x^2 + 2x &= 48 \\ x^2 + 2x - 48 &= 0 \\ (x+8)(x-6) &= 0 \\ x + 8 &= 0 \quad \text{or} \quad x - 6 = 0 \\ x &= -8 \quad \text{or} \quad x = 6 \end{aligned}$$

A garden cannot be -8 feet long, so we must use only the 6 as a value for x.

So, the width of the garden is 6 feet and the length is 8 feet.

Practice

Solve each problem. Refer to pages 466 and 467 as needed.

1. The product of two consecutive positive integers is 72. Find the integers.

   Answer: _____

2. The product of two consecutive positive integers is 90. Find the integers.

   Answer: _____

3. The product of two consecutive **negative odd integers** is 35. Find the integers.

   Answer: _____

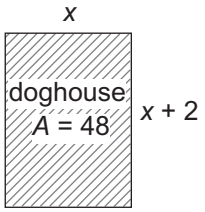
4. The product of two consecutive *negative odd integers* is 143. Find the integers.

   Answer: _____

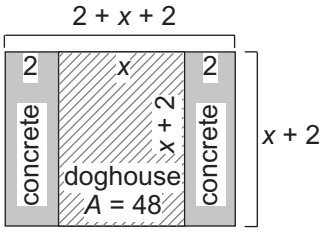
5. Sara has a photo that is 5 inches by 7 inches. When she adds a frame to it, she gets an area of 63 square inches. How wide is the frame?

   Answer: _____ inches

6. Bob wants to build a doghouse that is 2 feet longer than it is wide. He's building it on a concrete slab that will leave 2 feet of concrete slab visible on each side. If the area of the doghouse is 48 square feet, what are the dimensions of the concrete slab?



   Answer: _____ feet x _____ feet



7. Marianne has a scarf that is 5 inches longer than it is wide. If the area of her scarf is 84 square inches, what are the dimensions of her scarf?

   Answer: _____ inches x _____ inches

Practice

Use the list below to complete the following statements.

area (*A*)	**length (*l*)**	**positive integers**
consecutive integers	**negative integers**	**rectangle**
	odd integers	**width (*w*)**

1. _____ are integers that are less than zero.

2. Integers that are greater than zero are _____ .

3. _____ are *not* divisible by 2.

4. The measure in square units of the inside region of a two-dimensional figure is called the _____ .

5. A parallelogram with four right angles is called a(n) _____ .

6. _____ means the numbers are in order.

7. To find the area (*A*) of a rectangle, you multiply the _____ by the _____ .

8. _____ are the numbers in the set {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}.

Lesson Two Purpose

- Use systems of equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

Systems of Equations

When we look at an equation like $x + y = 5$, we see that because there are two **variables**, there are many possible solutions. For instance,

- if $x = 5$, then $y = 0$
- if $x = 2$, then $y = 3$
- if $x = -4$, then $y = 9$
- if $x = 2.5$, then $y = 2.5$, etc.

Another equation such as $x - y = 1$ allows a specific solution to be determined. Taken together, these two equations help to limit the possible solutions.

When taken together, we call this a **system of equations**. A *system of equations* is a group of two or more equations that are related to the same situation and share the same variables. Look at the equations below.

$$x + y = 5$$
$$x - y = 1$$

One possible way to solve the system of equations above is to **graph each equation** on the same set of **axes**. Use a **table** *of values* like those on the following page to help determine two possible **points** for each **line (↔)**.

Table of Values

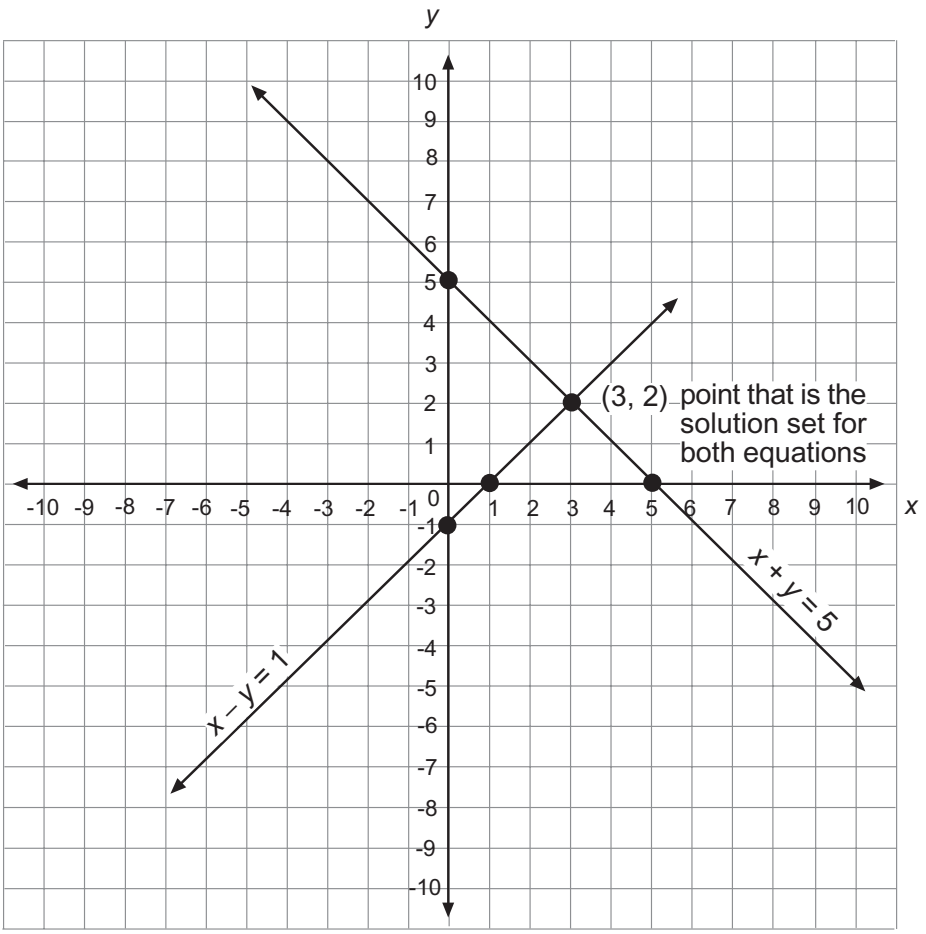
$x + y = 5$	
x	y
0	5
5	0

Table of Values

$x - y = 1$	
x	y
0	-1
1	0

Plot the *points* for the first equation on the **coordinate grid or plane** below, then draw a *line* connecting them. Do the same for the second set of points.

Graph of $x + y = 5$ and $x - y = 1$



We see from the **graph** above that the two lines **intersect** or *cross at a point*. That point (3, 2) is the solution set for both equations. It is the only point that makes both equations true. You can check your work by replacing x with 3 and y with 2 in both equations to see if they produce true statements.

Although graphing is one way to deal with systems of equations, it is not always the most accurate method. If our graph paper is not perfect, our pencil is not super-sharp, or the point of intersection is not at a corner on the grid, we may not get the correct answer.

The system can also be solved algebraically. Let's see how that works.

We know from past experience that we can solve problems more easily when there is only one *variable*. So, our job is to eliminate a variable. If we look at the two equations **vertically** (straight up and down), we see that by adding in columns, the y's will disappear.

$$\begin{aligned} x + y &= 5 \\ x - y &= 1 \\ \underline{2x + 0} &\underline{= 6} \end{aligned}$$

This leaves us with a new equation to solve:

$$\begin{aligned} 2x + 0 &= 6 \\ 2x &= 6 \\ \tfrac{2x}{2} &= \tfrac{6}{2} \quad \longleftarrow \text{divide both sides by 2} \\ x &= 3 \end{aligned}$$

We've found the value for x; now we must find the value of y. Use either of the original equations and replace the x with 3. The example below uses the first one.

$$\begin{aligned} x + y &= 5 \\ 3 + y &= 5 \\ 3 - 3 + y &= 5 - 3 \quad \longleftarrow \text{subtract 3 from both sides} \\ y &= 2 \end{aligned}$$

So, our solution set is {3, 2}.

Let's try another! We'll solve and then graph this time.

$$\begin{array}{rcr} 2x + y &=& 6 \\ -2x + 2y &=& -12 \\ \hline 0 + 3y &=& -6 \\ \frac{\cancel{3}y}{\cancel{3}} &=& \frac{-\cancel{6}}{\cancel{3}} = \frac{-2}{1} \\ y &=& -2 \end{array}$$

add to eliminate the *x*'s

solve

$$\begin{array}{rcl} 2x + -2 &=& 6 \\ 2x + -2 &=& 6 + 2 \\ 2x &=& 8 \\ \frac{\cancel{2}x}{\cancel{2}} &=& \frac{\cancel{8}}{\cancel{2}} = \frac{4}{1} \\ x &=& 4 \end{array}$$

replace *y* with -2 in one equation and solve for *x*

Our solution set is {4, -2}.

Now let's graph the two equations.

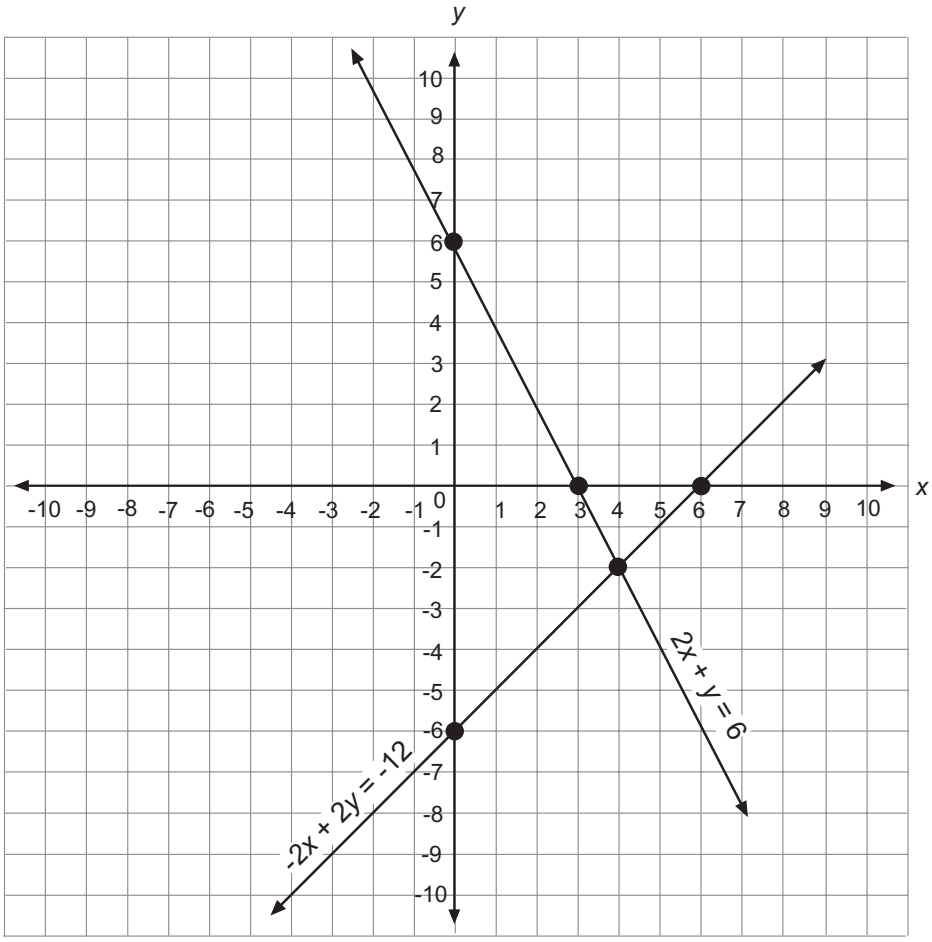
Table of Values

$2x + y = 6$	
x	y
0	6
3	0

Table of Values

$-2x + 2y = -12$	
x	y
0	-6
6	0

Graph of $2x + y = 6$ and $-2x + 2y = -12$



Note: Watch for these special situations.

- If the graphs of the equations are the same line, then the two equations are *equivalent* and have an **infinite**, (that is, there is *no* limit) to the number of possible solutions.

- If the graphs do not *intersect* at all, they are **parallel (||)**, and are an equal distance at every point, and have *no* possible solutions. The solution set would be empty—{ }.

Practice

Solve each **system of equations** *algebraically. Use the* **table of values** *to solve and* **graph** *both equations on the graphs provided. Refer to pages 461-465 as needed.*

Hint: *Two of the following sets of equations are* **equivalent expressions** *and will have the same line with an* infinite *number of possible solutions. See note on the previous page.*

1. $x - y = -1$
   $x + y = 7$

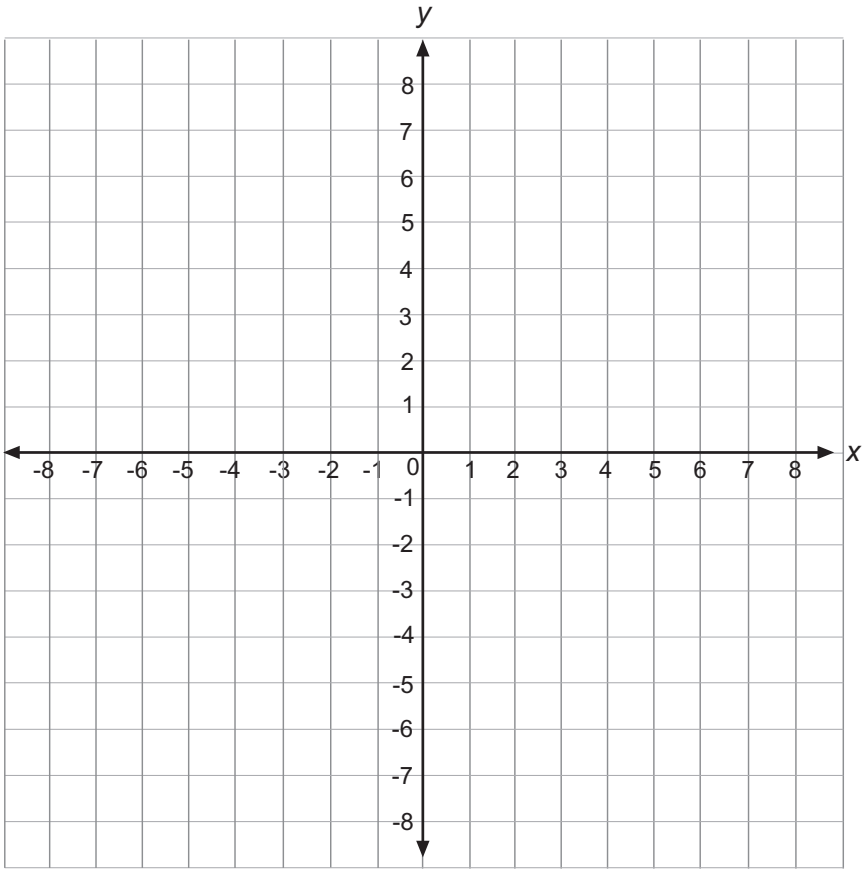
Table of Values

$x - y = -1$	
x	y

Table of Values

$x + y = 7$	
x	y

Graph of $x - y = -1$ and $x + y = 7$



2. $2x - y = 4$
$x + y = 5$

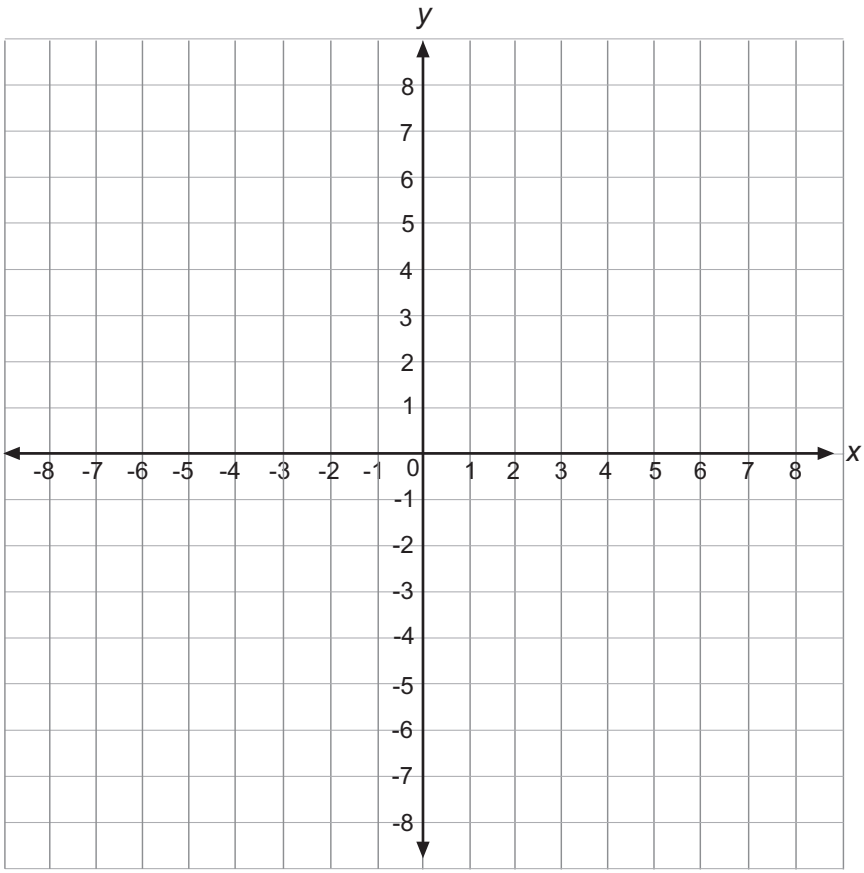
Table of Values

$2x - y = 4$	
x	y

Table of Values

$x + y = 5$	
x	y

Graph of $2x - y = 4$ and $x + y = 5$



3. $4x - y = 2$
$-2x + y = 0$

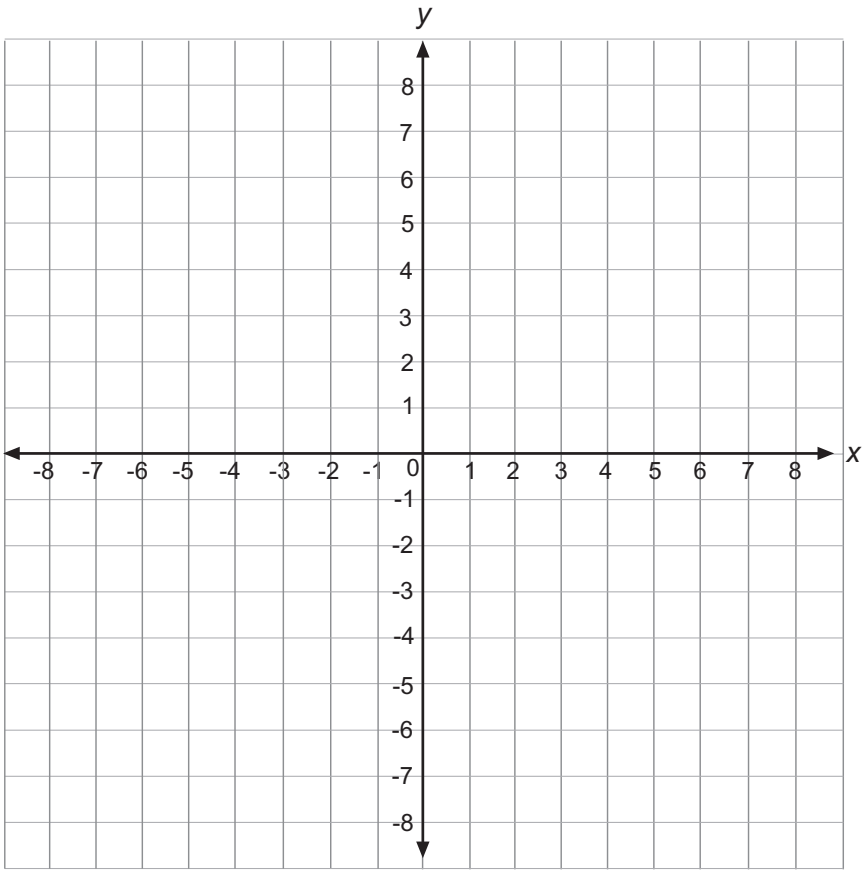
Table of Values

$4x - y = 2$	
x	y

Table of Values

$-2x + y = 0$	
x	y

Graph of $4x - y = 2$ and $-2x + y = 0$



4. $x - 2y = 4$
$2x - 4y = 8$

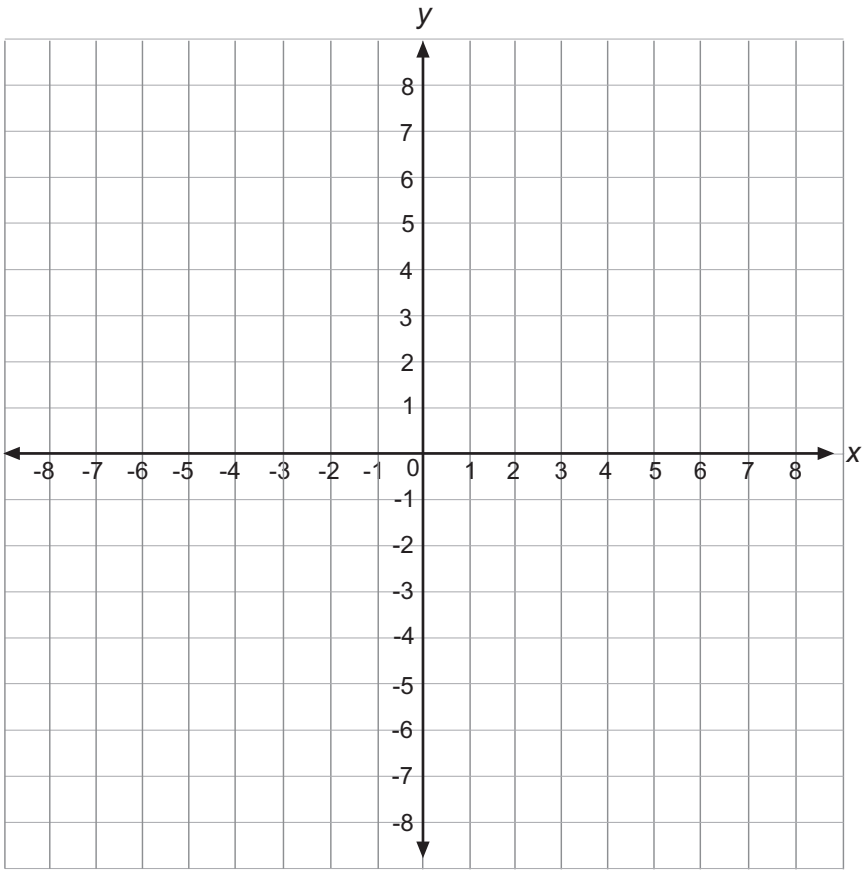
Table of Values

$x - 2y = 4$	
x	y

Table of Values

$2x - 4y = 8$	
x	y

Graph of $x - 2y = 4$ and $2x - 4y = 8$



5. $2x + y = 8$
$-2x + y = -4$

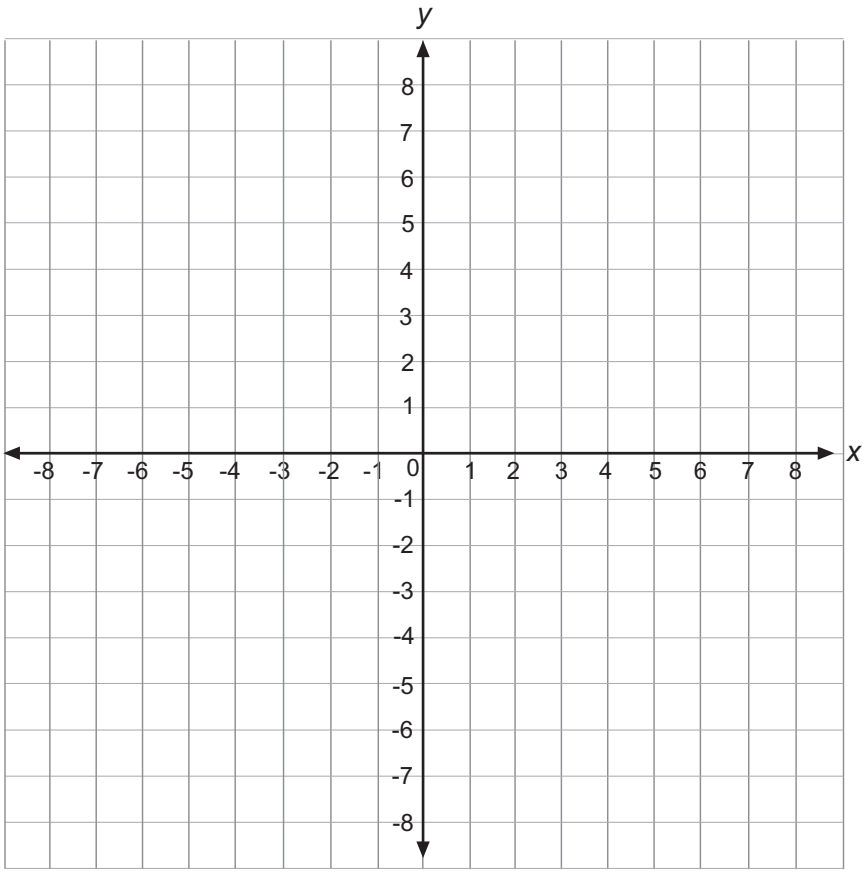
Table of Values

$2x + y = 8$	
x	y

Table of Values

$-2x + y = -4$	
x	y

Graph of $2x + y = 8$ and $-2x + y = -4$



6. $3x - 2y = -1$
$-6x + 4y = 2$

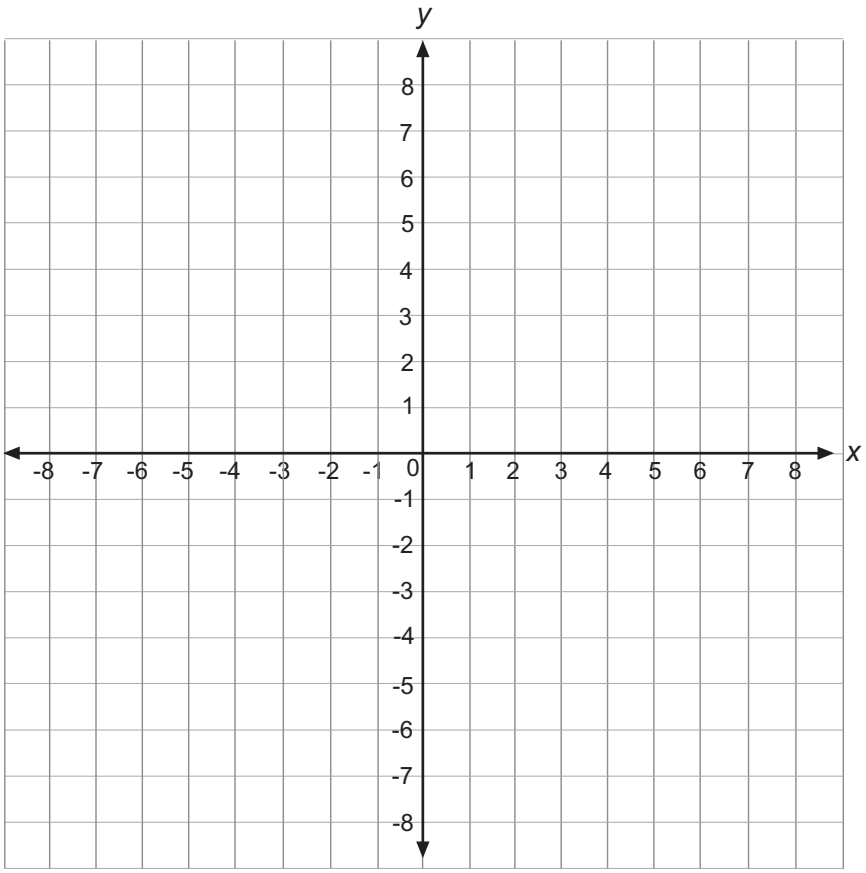
Table of Values

$3x - 2y = -1$	
x	y

Table of Values

$-6x + 4y = 2$	
x	y

Graph of $3x - 2y = -1$ and $-6x + 4y = 2$



Using Substitution to Solve Equations

There are other processes we can use to solve systems of equations. Let's take a look at some of the options.

Example 1

Suppose our two equations are as follows.

$$\begin{aligned} 2x + 3y &= 14 \\ x &= 4 \end{aligned}$$

To solve this system, we could more easily use a method called **substitution**. We simply put the value of x from the second equation in for the x in the first equation.

$$\begin{aligned} 2x + 3y &= 14 && \leftarrow \textbf{substitute } 4 \text{ for } x \\ 2(4) + 3y &= 14 && \leftarrow \textbf{simplify} \\ 8 + 3y &= 14 \\ 8 - 8 + 3y &= 14 - 8 && \leftarrow \text{subtract} \\ 3y &= 6 \\ \frac{3y}{3} &= \frac{6}{3} && \leftarrow \text{divide} \\ y &= 2 \end{aligned}$$

The solution set is {4, 2}.

Example 2

This one is a little more complex.

Below are our two equations.

$$4x - y = -2$$
$$x = y + 4$$

We can *substitute* $(y + 4)$ from the second equation in for x in the first equation.

$$4x - y = -2 \quad \text{substitute } (y + 4) \text{ for } x$$
$$4(y + 4) - y = -2 \quad \textbf{distribute}$$
$$4y + 16 - y = -2 \quad \text{simplify}$$
$$3y + 16 = -2 \quad \text{subtract}$$
$$3y = -18 \quad \text{divide}$$
$$y = -6$$

Notice that $(y + 4)$ is in parentheses. This helps us remember to *distribute* when the time comes.

Now we must find the value of x. Use an original equation and substitute -6 for y and then solve for x.

$$4x - y = -2 \quad \text{original equation}$$
$$4x - (-6) = -2 \quad \text{substitute } (-6) \text{ for } y$$
$$4x + 6 = -2 \quad \text{simplify}$$
$$4x = -8 \quad \text{subtract}$$
$$x = -2 \quad \text{divide}$$

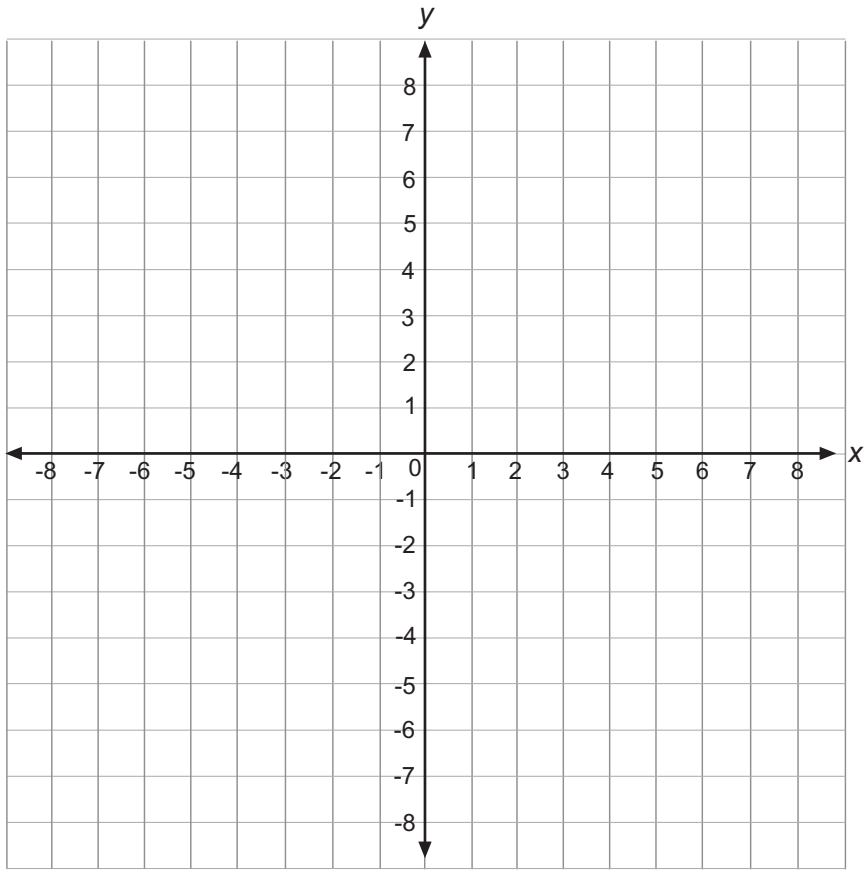
Now try the practice on the following page.

Practice

Solve each **system of equations** *algebraically. Use the* **substitution method** *to solve and* **graph** *both equations on the graphs provided. Refer to pages 482 and 483 as needed.*

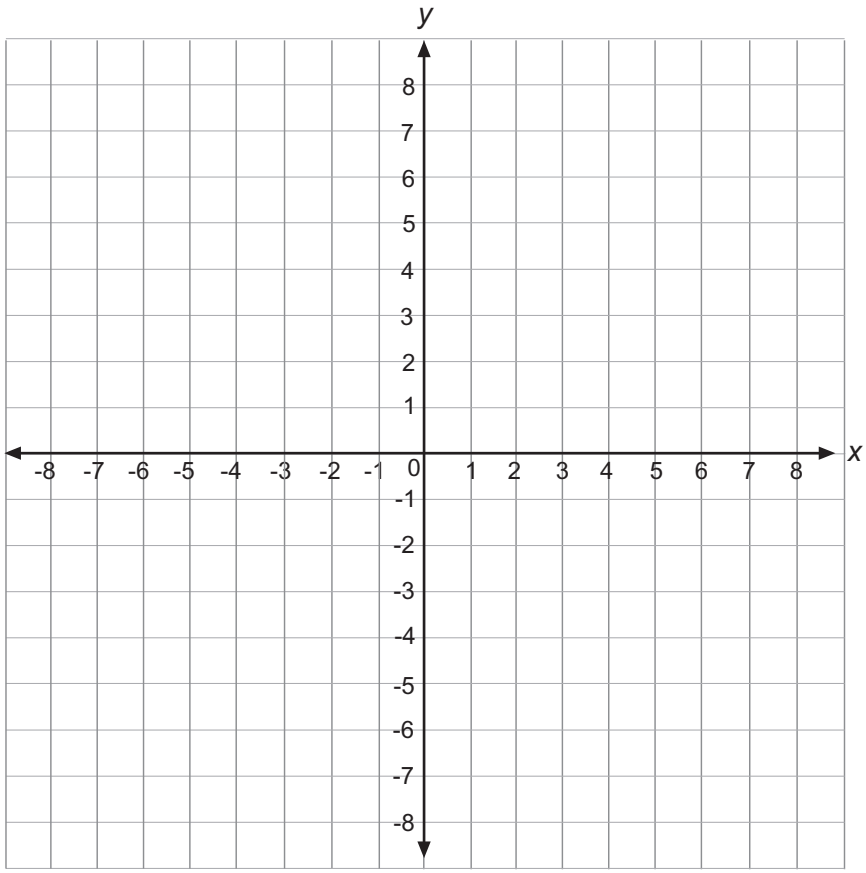
1. $3x - 2y = 6$
   $x = 4$

Graph of $3x - 2y = 6$ and $x = 4$



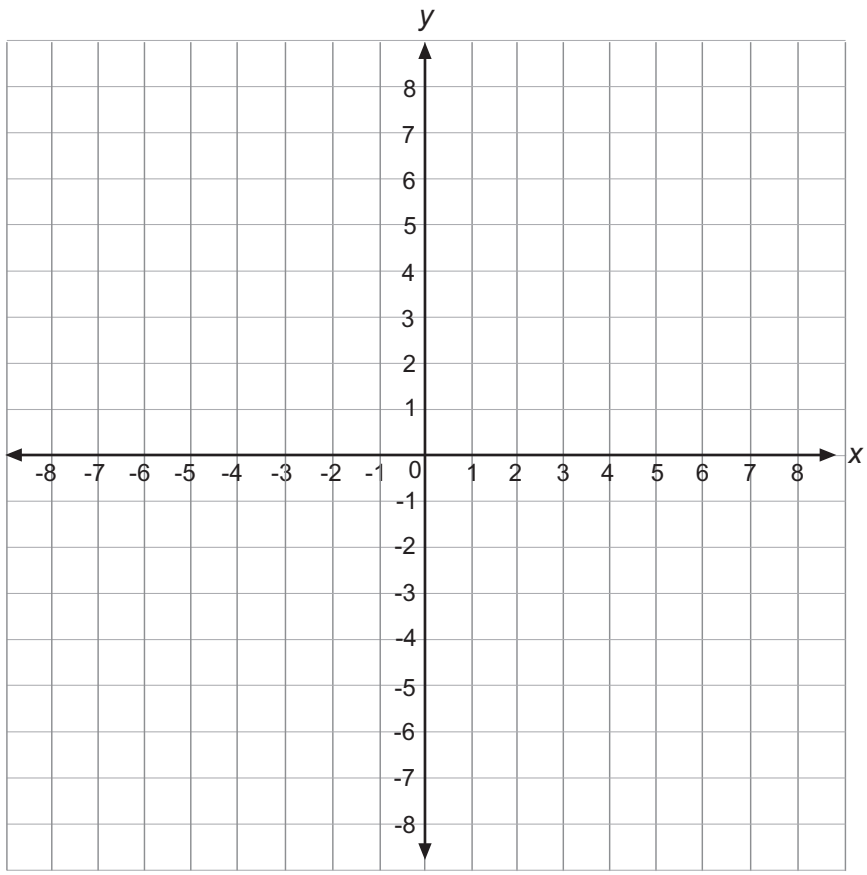
2. $5x - y = 9$
$x = 2y$

Graph of $5x - y = 9$ and $x = 2y$



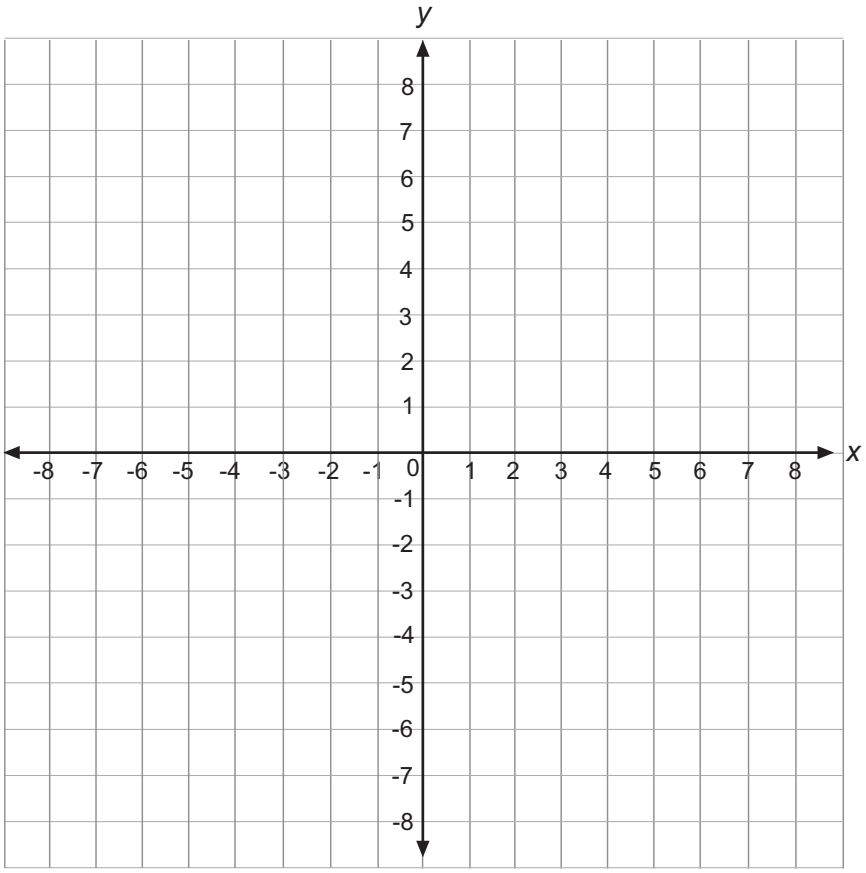
3. $x + y = 5$
$x = y + 1$

Graph of $x + y = 5$ and $x = y + 1$



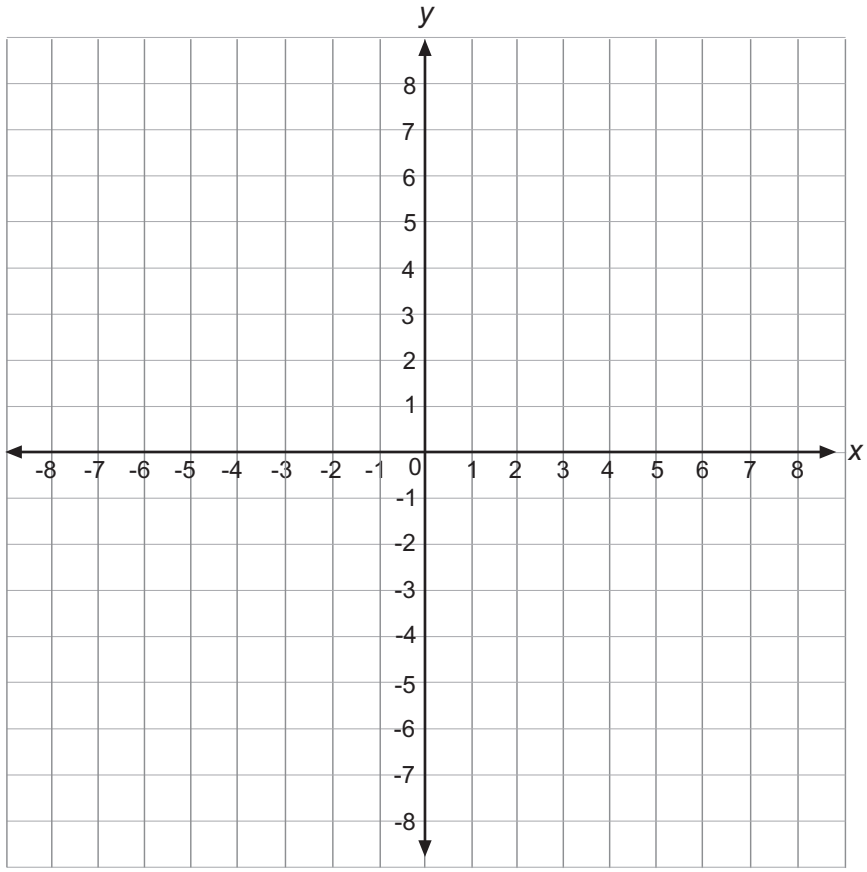
4. $5x + y = -15$
$y = 1 - x$

Graph of $5x + y = -15$ and $y = 1 - x$



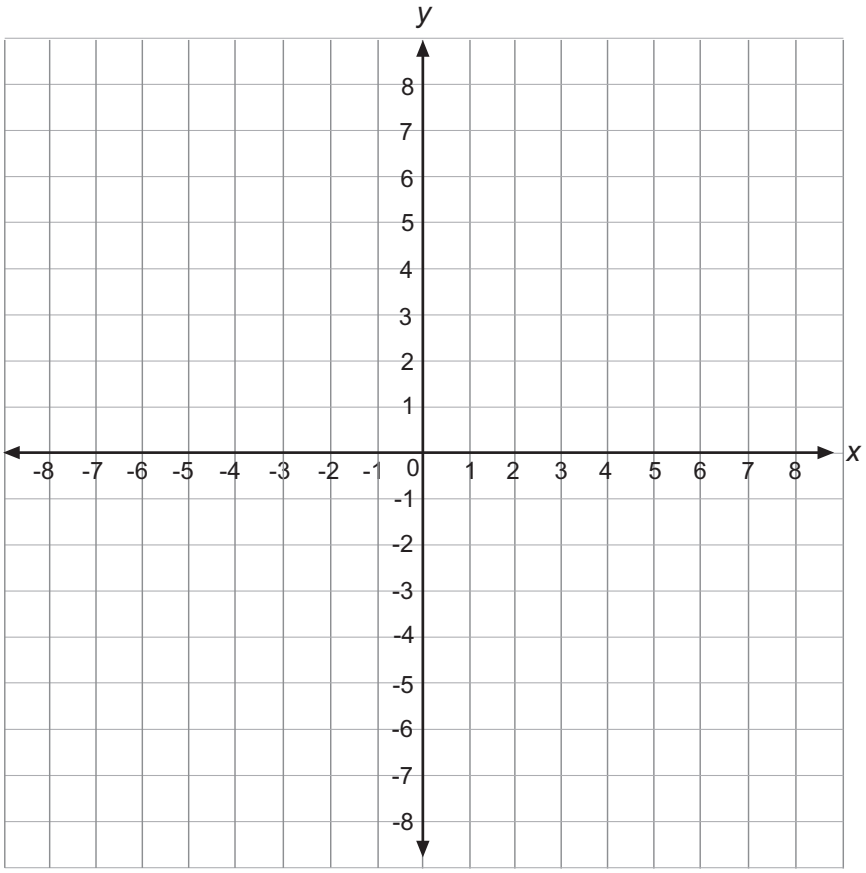
5. $$\begin{aligned} x &= 2y + 15 \\ 4x + 2y &= 10 \end{aligned}$$

Graph of $x = 2y + 15$ and $4x + 2y = 10$



6. $x + 2y = 14$
$x = 3y - 11$

Graph of $x + 2y = 14$ and $x = 3y - 11$



Using Magic to Solve Equations

There are times when neither the algebraic or substitution method seems like a good option. If the equations should look similar to these, we have another option.

Example 1

$$\begin{aligned} 5x + 12y &= 41 \\ 9x + 4y &= 21 \end{aligned}$$

We have to perform a little "math-magic" to solve this problem. When looking at these equations, you should see that if the $4y$ were $-12y$ instead, we could add vertically and the y's would disappear from the equation.

So, our job is to make that $4y$ into $-12y$. We could do that by multiplying $4y$ by -3. The only catch is that we must multiply the whole equation by -3 to keep everything balanced.

$$\begin{aligned} 9x + 4y &= 21 && \leftarrow \text{original equation} \\ -3(9x + 4y &= 21) && \leftarrow \text{multiply equation by -3} \\ -27x + (-12y) &= -63 && \leftarrow \text{new 2nd equation} \end{aligned}$$

Now line up the equations, replacing the second one with the new equation.

$$\begin{aligned} 5x + 12y &= 41 && \leftarrow \text{original 1st equation} \\ -27x + (-12y) &= -63 && \leftarrow \text{new 2nd equation} \\ \hline -22x + 0 &= -22 && \leftarrow \text{subtract vertically} \\ -22x &= -22 && \leftarrow \text{simplify} \\ x &= 1 && \leftarrow \text{divide} \end{aligned}$$

Now that we know the value of x, we can replace x with 1 in the original equation and solve for y.

$$\begin{aligned} 5x + 12y &= 41 && \leftarrow \text{original 1st equation} \\ 5(1) + 12y &= 41 && \leftarrow \text{substitute (1) for } x \\ 5 + 12y &= 41 && \leftarrow \text{simplify} \\ 12y &= 36 && \leftarrow \text{subtract} \\ y &= 3 && \leftarrow \text{divide} \end{aligned}$$

Our solution set is {1, 3}. Be sure to put the answers in the *correct order* because they are an **ordered pair**, where the first and second value represent a position on a *coordinate grid or system*.

Sometimes you may have to perform "math-magic" on both equations to get numbers to "disappear."

Example 2

$$\begin{aligned} 3x - 4y &= 2 \\ 2x + 3y &= 7 \end{aligned}$$

After close inspection, we see that this will take double magic. If the **coefficients** of the x's could be made into a $6x$ and a $-6x$, this problem might be solvable. Let's try!

Multiply the first equation by 2 and the second equation by -3.

$$\begin{array}{rcrl} 2(3x - 4y = 2) & \longrightarrow & 6x - 8y = 4 & \longleftarrow \text{1st equation} \bullet 2 \\ -3(2x + 3y = 7) & \longrightarrow & -6x - 9y = -21 & \longleftarrow \text{2nd equation} \bullet -3 \\ \hline & & 0 - 17y = -17 & \\ & & -17y = -17 & \\ & & y = 1 & \end{array}$$

Use $y = 1$ to find the value of x using an original equation.

$$\begin{aligned} 3x - 4y &= 2 && \longleftarrow \text{original 1st equation} \\ 3x - 4(1) &= 2 && \longleftarrow \text{substitute (1) for } y \\ 3x - 4 &= 2 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

The solution set is {2, 1}.

Now it's your turn to practice on the next page.

Practice

Solve each of the following **systems of equations**. *Refer to pages 490 and 491 as needed.*

1. $$\begin{aligned} 3x + y &= 7 \\ 2x - 3y &= 12 \end{aligned}$$

2. $$\begin{aligned} 3x + y &= 11 \\ x + 2y &= 12 \end{aligned}$$

3. $$\begin{aligned} 9x + 8y &= -45 \\ 6x + y &= 9 \end{aligned}$$

4. $$\begin{aligned} -5x + 4y &= 4 \\ 4x - 7y &= 12 \end{aligned}$$

5. $$\begin{aligned} -2x - 11y &= 4 \\ 5x + 9y &= 27 \end{aligned}$$

6. $$\begin{aligned} 2x + 3y &= 20 \\ 3x + 2y &= 15 \end{aligned}$$

Solving More Word Problems

Let's see how we might use the methods we've learned to solve word problems.

Example 1

Twice the **sum** of two integers is 20. The larger integer is 1 more than twice the smaller. Find the integers.

Let S = the small integer
Let L = the larger integer

Now, write equations to fit the wording in the problem.

$2(S + L) = 20$ and $L = 2S + 1$

$$
\begin{aligned}
2S + 2L &= 20 && \leftarrow \text{simplify} \\
2S + 2(2S + 1) &= 20 && \leftarrow \text{substitute } (2S + 1) \text{ for } L \\
2S + 4S + 2 &= 20 && \leftarrow \text{distribute} \\
6S + 2 &= 20 && \leftarrow \text{simplify} \\
6S &= 18 && \leftarrow \text{subtract} \\
S &= 3 && \leftarrow \text{divide}
\end{aligned}
$$

Since the smaller integer is 3, the larger one is 2(3) + 1 or 7. The integers are 3 and 7.

Example 2

Three tennis lessons and three golf lessons cost \$60. Nine tennis lessons and six golf lessons cost \$147. Find the cost of one tennis lesson and one golf lesson.

Let T = the cost of 1 tennis lesson
Let G = the cost of 1 golf lesson

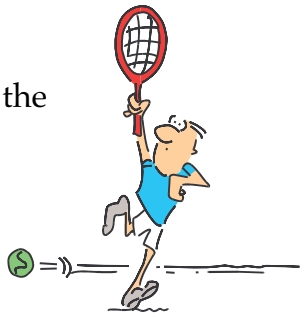
Use the variables to interpret the sentences and make equations.

$$\begin{aligned} 3T + 3G &= 60 \\ 9T + 6G &= 147 \end{aligned}$$

Make the *coefficients* of G match by multiplying the first equation by -2.

$$\begin{aligned} -6T + -6G &= -120 && \leftarrow -2(3T + 3G = 60) \\ 9T + 6G &= 147 && \leftarrow \text{bring in the 2nd equation} \\ \hline 3T + 0 &= 27 && \leftarrow \text{subtract} \\ 3T &= 27 && \leftarrow \text{divide} \\ T &= 9 \end{aligned}$$

We know that one tennis lesson costs \$9, so let's find the cost of one golf lesson.



$$\begin{aligned} 3T + 3G &= 60 && \leftarrow \text{original 1st equation} \\ 3(9) + 3G &= 60 && \leftarrow \text{substitute (9) for } T \\ 27 + 3G &= 60 && \leftarrow \text{simplify} \\ 3G &= 33 && \leftarrow \text{divide} \\ G &= 11 \end{aligned}$$

So, one tennis lesson costs \$9 and one golf lesson costs \$11.

Now you try a few items on the next page.

Practice

Solve each of the following. Refer to pages 494 and 495 as needed.

1. The sum of two numbers is 35. The larger one is 4 times the smaller one. Find the two numbers.

   Answer: _____ and _____

2. A 90-foot cable is cut into two pieces. One piece is 18 feet longer than the shorter one. Find the lengths of the two pieces.

   Answer: _____ feet and _____ feet

3. Joey spent $98 on a pair of jeans and a shirt. The jeans cost $20 more than the shirt. How much did each cost?

   Answer: jeans = $ _____ and shirt = $ _____

4. Four cheeseburgers and three drinks cost $13. Two drinks cost $0.60 more than one cheeseburger. Find the cost of one drink and one cheeseburger.

   Answer: $ _____

5. The football team at Leon High School has 7 more members than the team from Central High School. Together the two teams have 83 players. How many players does each team have?

   Answer: Central = _____ and Leon = _____

6. Andre earns $40 a week less than Sylvia. Together they earn $360 each week. How much does each earn?

   Answer: Sylvia = $ _____ and Andre = $ _____

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. a two-dimensional network of horizontal and vertical lines that are parallel and evenly-spaced

_____ 2. all points whose coordinates are solutions of an equation

_____ 3. a group of two or more equations that are related to the same situation and share variables

_____ 4. a drawing used to represent data

_____ 5. at right angles to the horizon; straight up and down

_____ 6. any symbol, usually a letter, which could represent a number

_____ 7. the horizontal and vertical number lines used in a coordinate plane system

_____ 8. a data display that organizes information about a topic into categories

_____ 9. to meet or cross at one point

A. axes (of a graph)

B. coordinate grid or plane

C. graph

D. graph of an equation

E. intersect

F. system of equations

G. table (or chart)

H. variable

I. vertical

Practice

Use the list below to write the correct term for each definition on the line provided.

coefficient	**parallel (∥)**	**substitution**
infinite	**simplify an expression**	**sum**
ordered pair	**substitute**	

_____ 1. a method used to solve a system of equations in which variables are replaced with known values or algebraic expressions

_____ 2. the result of adding numbers together

_____ 3. the location of a single point on a rectangular coordinate system where the first and second values represent the position relative to the x-axis and y-axis, respectively

_____ 4. to perform as many of the indicated operations as possible

_____ 5. to replace a variable with a numeral

_____ 6. having no boundaries or limits

_____ 7. being an equal distance at every point so as to never intersect

_____ 8. the number part in front of an algebraic term signifying multiplication

Lesson Three Purpose

- Use systems of equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

Graphing Inequalities

When graphing **inequalities**, you use much the same processes you used when graphing equations. The difference is that *inequalities* give you infinitely larger sets of solutions. In addition, your results with inequalities are always expressed using the following terms in relation to another expression:

- greater than ($>$)
- greater than or equal to ($\geq$)
- less than ($<$)
- less than or equal to ($\leq$)
- not equal to ($\neq$).

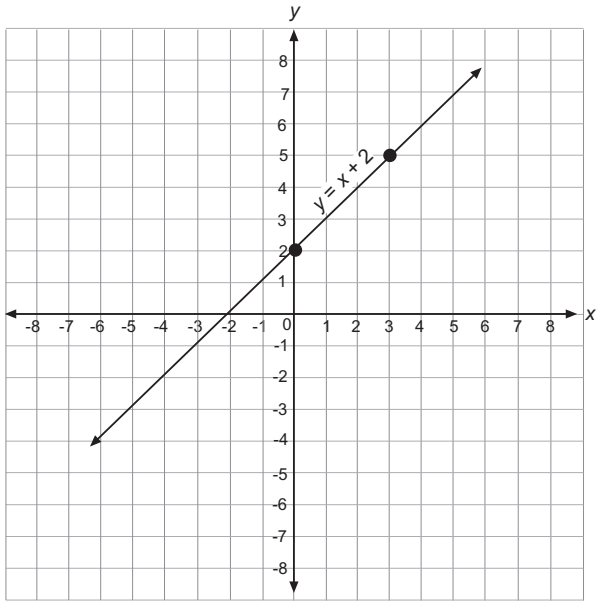
Therefore, we cannot graph an inequality as a line or a point. We must illustrate the entire set of answers by *shading* our graphs.

For instance, when we graph $y = x + 2$ using points, we found by using the table of values below, we get the line seen in Graph 1 below.

Table of Values

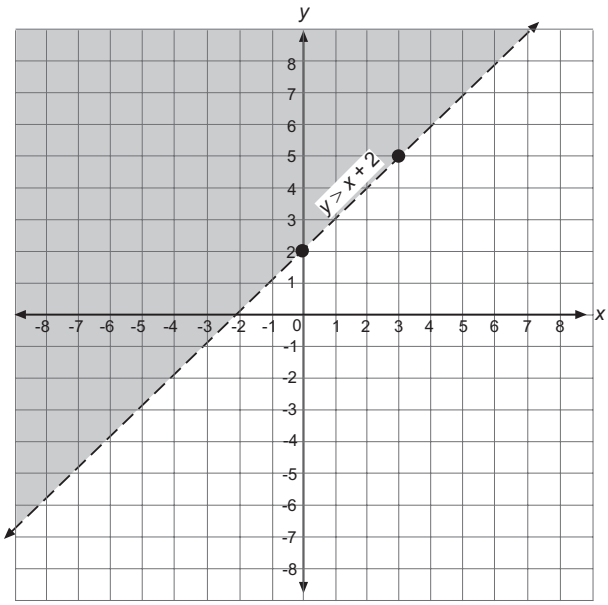
$y = x + 2$	
x	y
0	2
3	5

Graph 1 of $y = x + 2$



But when we graph $y > x + 2$, we use the line we found in Graph 1 as a *boundary*. Since $y \neq x + 2$, we show that by making the boundary line *dotted* (←- - -→). Then we shade the appropriate part of the grid. Because this is a "greater than" (>) problem, we shade *above* the dotted boundary line. See Graph 2 below.

Graph 2 of $y > x + 2$

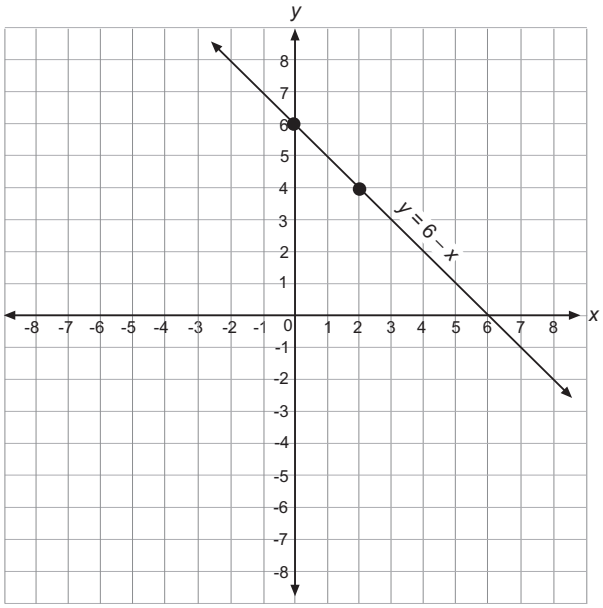


Suppose we wanted to graph $x + y \leq 6$. We first transform the inequality so that y is alone on the left side: $y \leq 6 - x$. We find a pair of points using a table of values, then graph the boundary line. Use the equation $y = 6 - x$ to find two pairs of points in the table of values. Graph the line that goes through points (0, 6) and (2, 4) from the table of values.

Table of Values

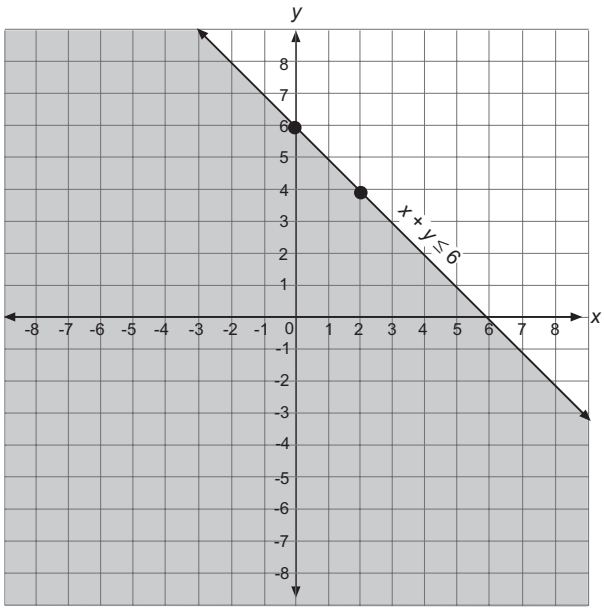
$y = 6 - x$	
x	y
0	6
2	4

Graph 3 of $y = 6 - x$



Now look at the inequality again. The symbol was ≤, so we leave the line solid and shade below the line.

Graph 4 of $x + y \leq 6$



Remember: *Change the inequality sign* whenever you *multiply* or *divide* the inequality by a **negative number**.

$$< \rightarrow >$$
$$> \rightarrow <$$
$$\leq \rightarrow \geq$$
$$\geq \rightarrow \leq$$

Inequality signs always change when multiplying or dividing by *negative numbers*.

Note:

- **Greater than** (>) means to shade **above** or to the **right** of the line.
- **Less than** (<) means to shade **below** or to the **left** of the line.

Test for Accuracy before You Shade

You can test your graph for accuracy before you shade by *choosing a point* that *satisfies the inequality*. Do *not* choose a point *on the boundary line*. *Choose a point* that falls in the area you are about to shade.

Practice

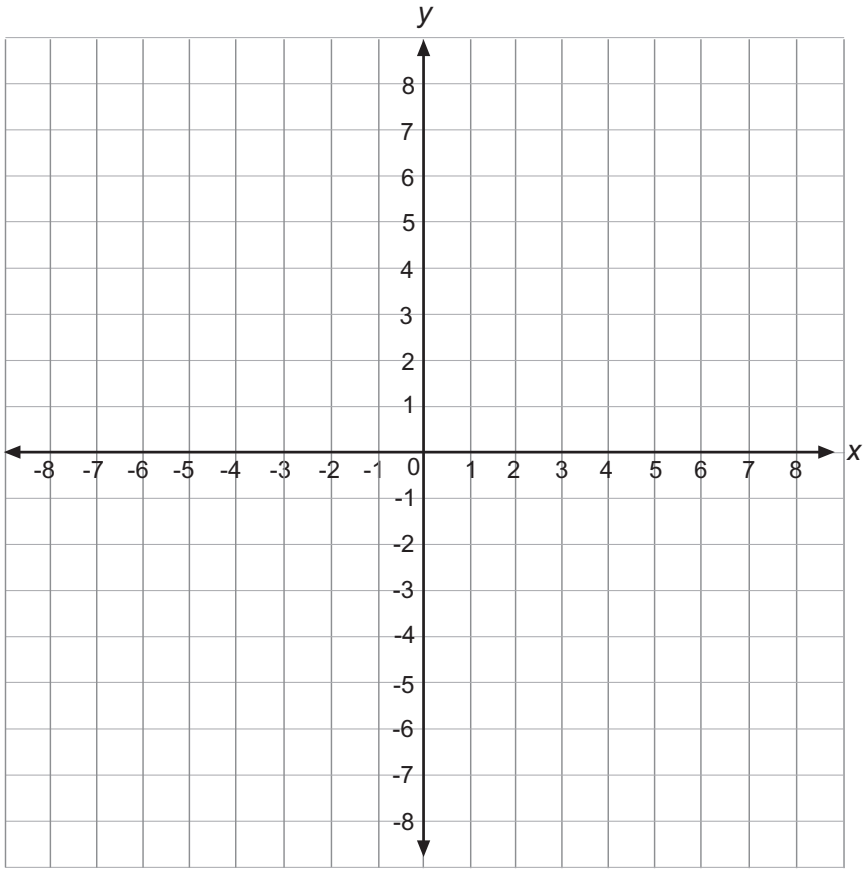
Graph each of the following **inequalities** *on the graphs provided. Refer to pages 500-503 as needed.*

1. $y \geq 2x - 3$

Table of Values

$y \geq 2x - 3$	
x	y

Graph of $y \geq 2x - 3$

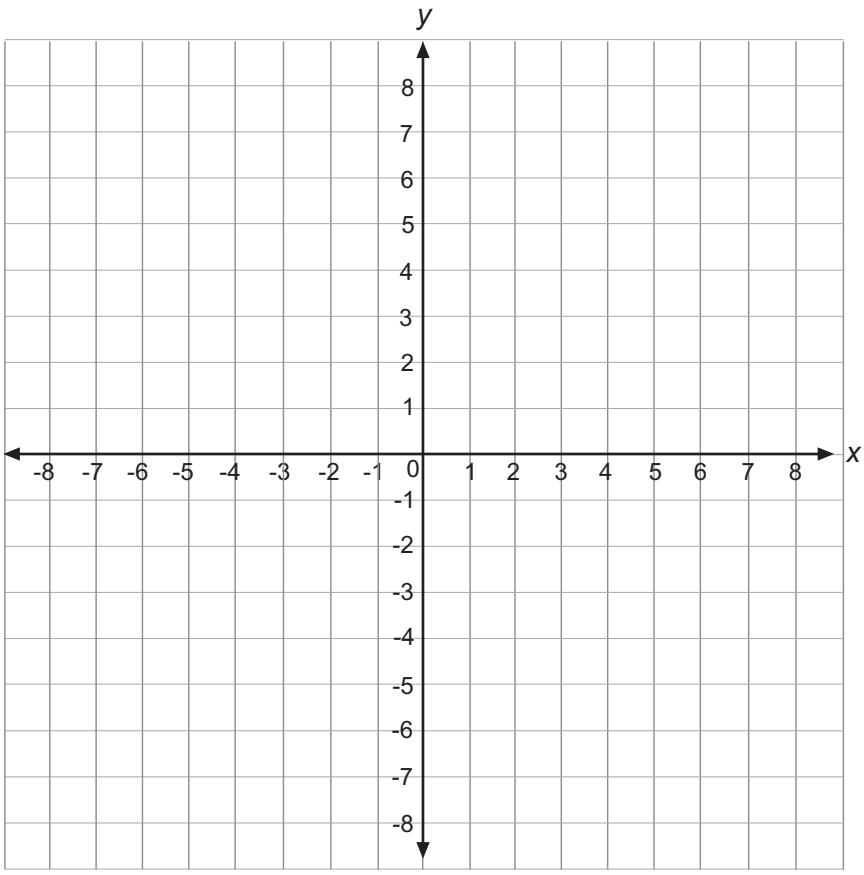


2. $y < x + 4$

Table of Values

$y < x + 4$	
x	y

Graph of $y < x + 4$

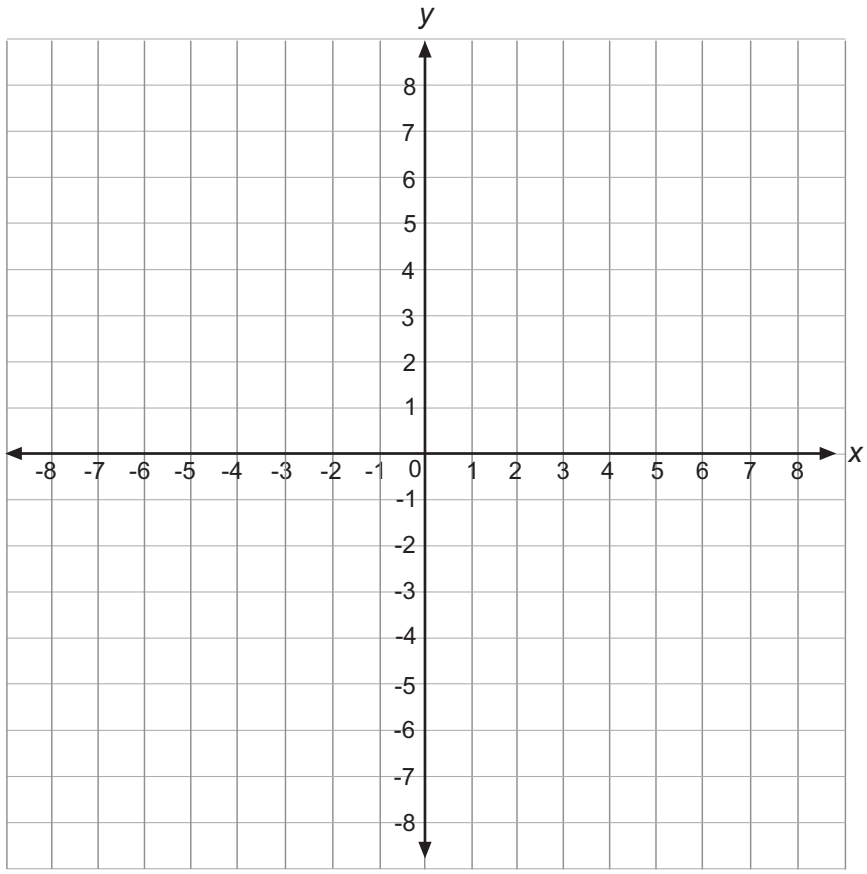


3. $y \leq 3x + 1$

Table of Values

$y \leq 3x + 1$	
x	y

Graph of $y \leq 3x + 1$

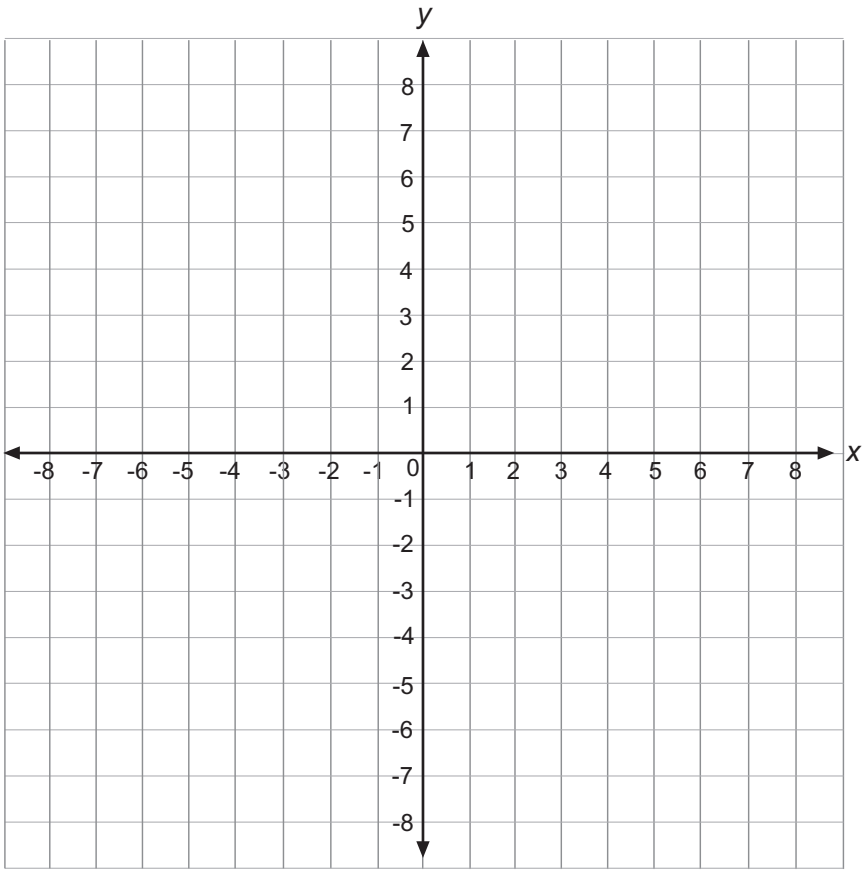


4. $y > x$

Table of Values

$y > x$	
x	y

Graph of $y > x$

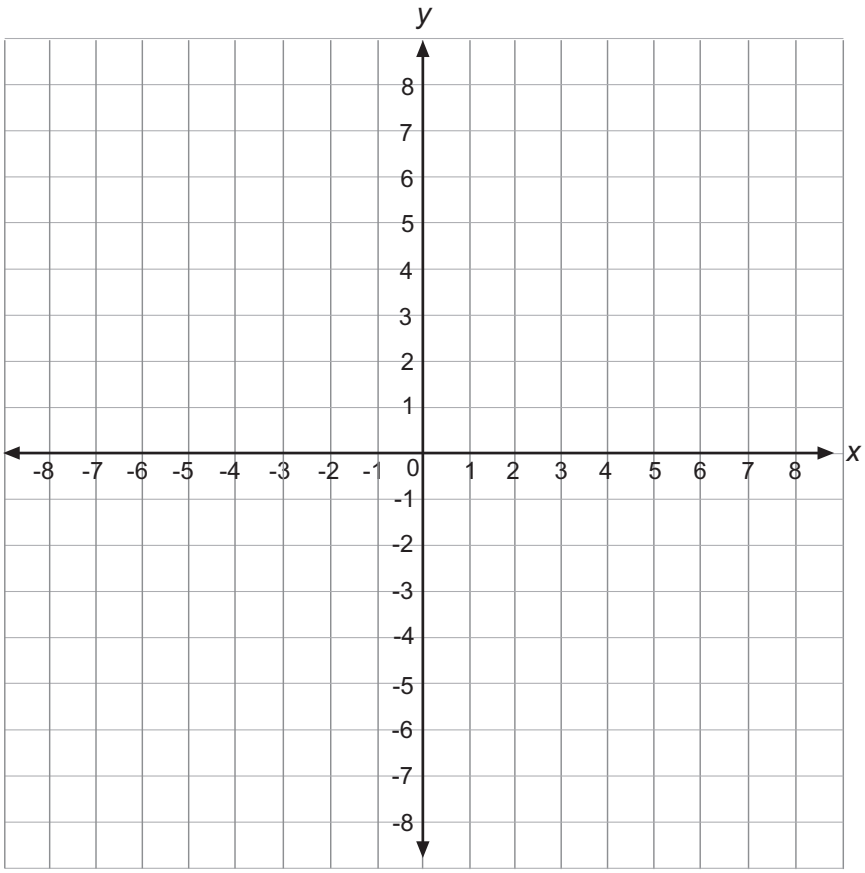


5. $x + y < -5$

Table of Values

$x + y < -5$	
x	y

Graph of $x + y < -5$

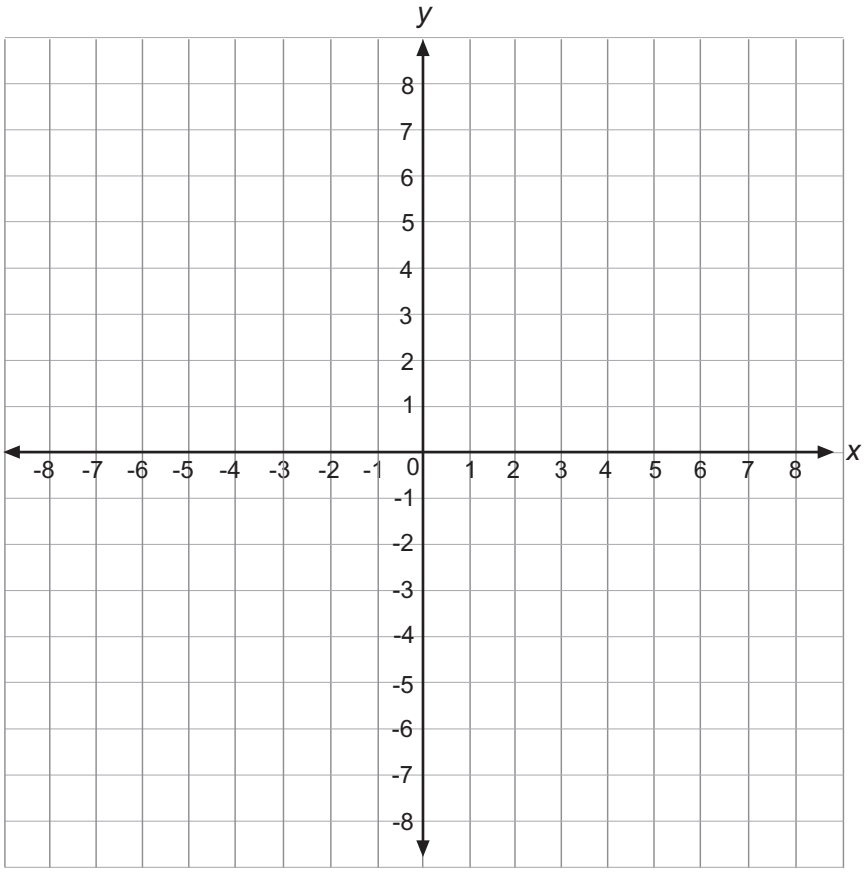


6. $x - 5y \geq 10$

Table of Values

$x - 5y \geq 10$	
x	y

Graph of $x - 5y \geq 10$

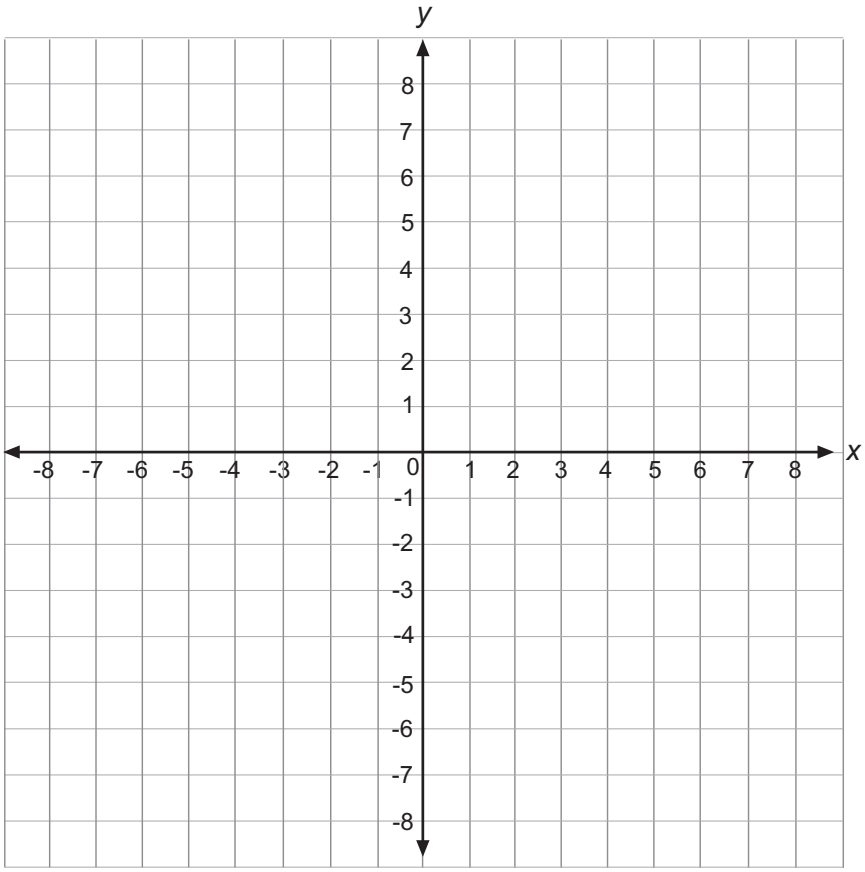


7. $x - 5y \leq 10$

Table of Values

$x - 5y \leq 10$	
x	y

Graph of $x - 5y \leq 10$

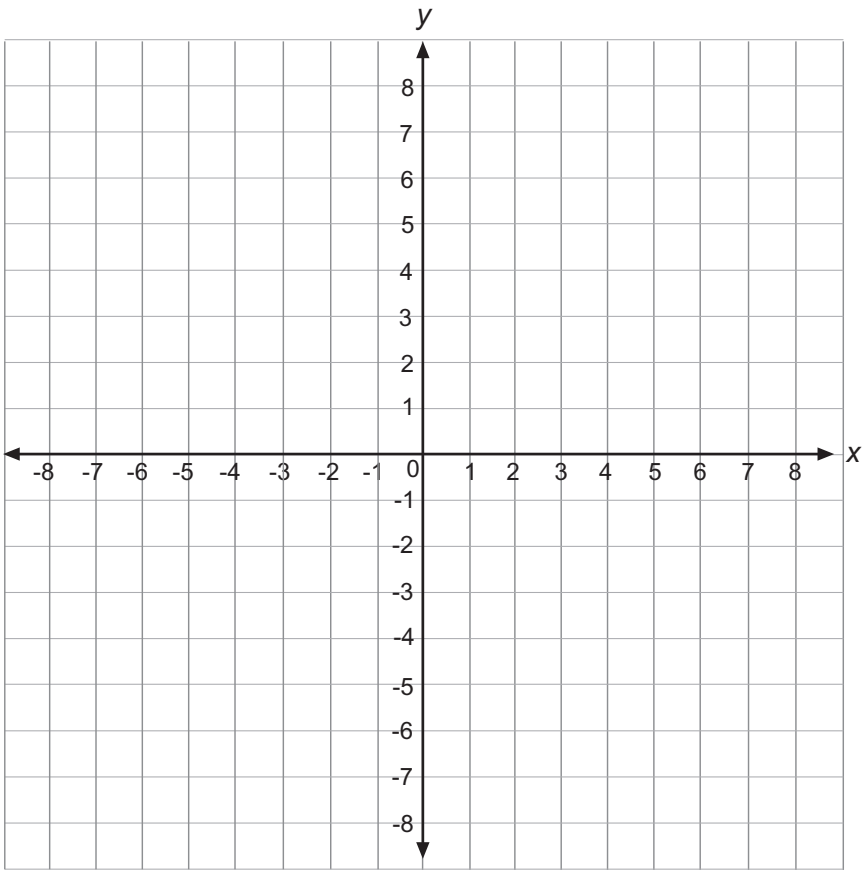


8. $y \leq -3$

Table of Values

$y \leq -3$	
x	***y***

Graph of $y \leq -3$



Graphing Multiple Inequalities

We can graph two or more inequalities on the same grid to find what solutions the two inequalities have in common or to find those solutions that work in one inequality or the other. The key words are "and" and "or." Let's see how these small, ordinary words affect our graphing.

Example 1

Graphically show the solutions for $2x + 3y > 6$ **and** $y \leq 2x$.

Note: See how the inequality $2x + 3y > 6$ is transformed in the table of values into the equivalent inequality $y > 2 - \frac{2}{3}x$. Refer to pages 500-503 as needed.

Step 1. Find the boundary lines for the two inequalities and draw them. Remember to make the line for the first inequality *dotted*.

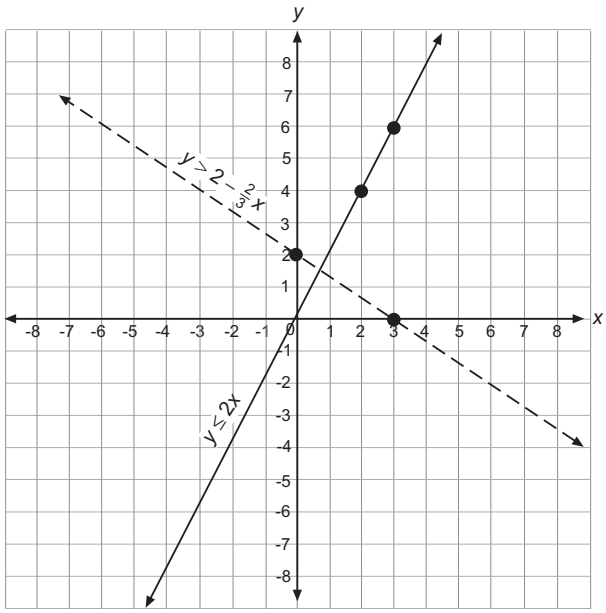
Table of Values

$y > 2 - \frac{2}{3}x$	
x	y
0	2
3	0

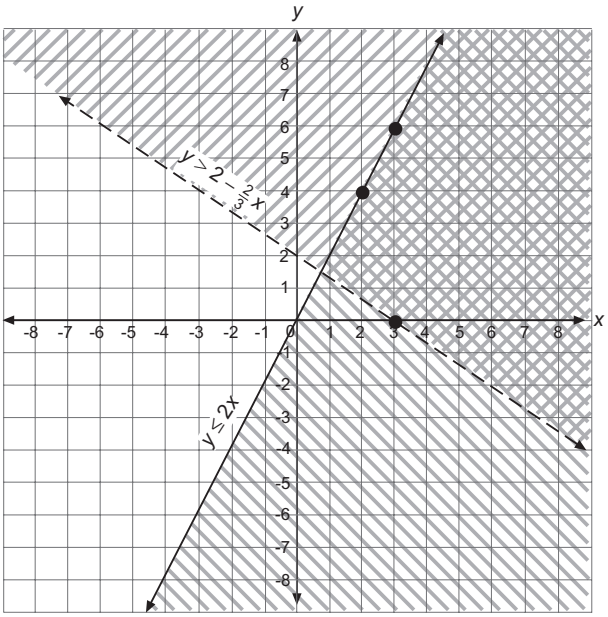
Table of Values

$y \leq 2x$	
x	y
3	6
2	4

Graph 5 of $2x + 3y > 6$ and $y \leq 2x$



Graph of $2x + 3y > 6$ and $y \leq 2x$ with Inequalities Shaded



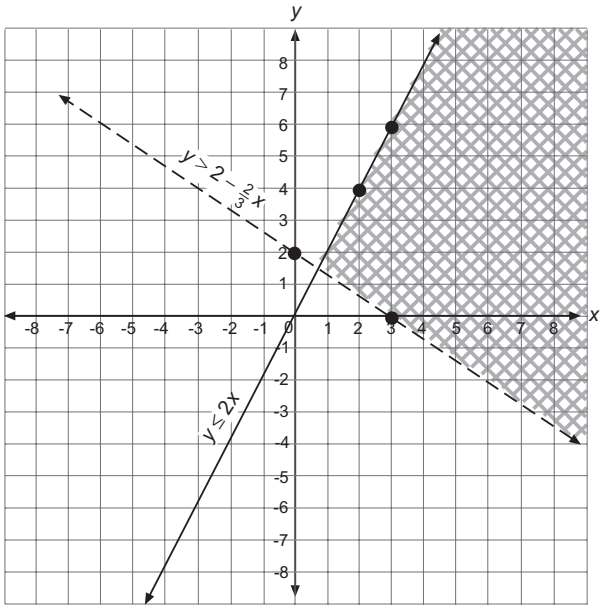
Step 2. Since the 1st inequality is *greater than*, shade *above* the dotted line.

Step 3. Shade the 2nd inequality *below* the solid line using a different type shading or different color.

Step 4. Because this was an "and" problem, this means we want to have as our solution *only* the parts where both shadings appear at the same time (in other words, where the shadings *overlap*). We want to show *only* those solutions that are *valid in both* inequalities *at the same time*.

Step 5. The solution for $2x + 3y > 6$ **and** $y \leq 2x$ is shown to the right.

Solution for Graph of $2x + 3y > 6$ and $y \leq 2x$



Example 2

Let's see how the graph of the solution would look if the problem had been $2x + 3y > 6$ **or** $y \leq 2x$.

We follow the same steps from 1 and 2 of the previous example.

Step 1. Find the boundary lines for the two inequalities and draw them. Remember to make the line for the first inequality *dotted*.

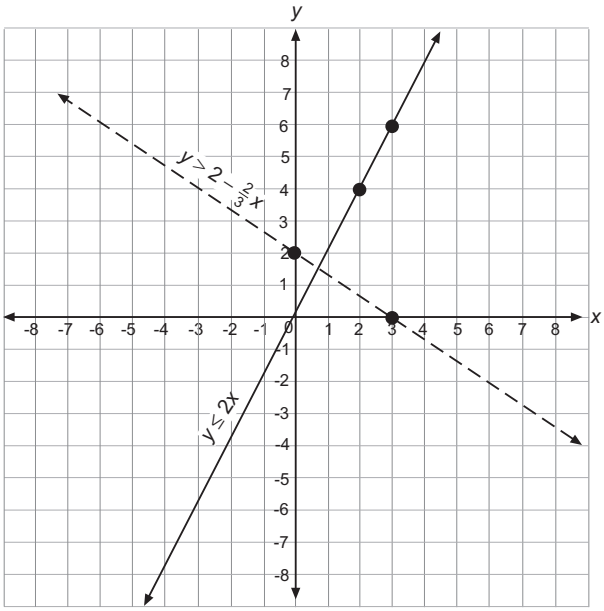
Table of Values

$y > 2 - \frac{2}{3}x$	
x	y
0	2
3	0

Table of Values

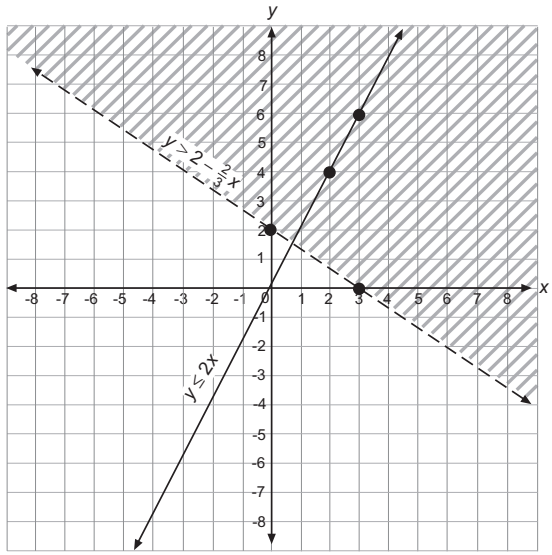
$y \leq 2x$	
x	y
3	6
2	4

Graph of $2x + 3y > 6$ or $y \leq 2x$



Step 2. Since the 1st inequality is *greater than*, shade *above* the dotted line.

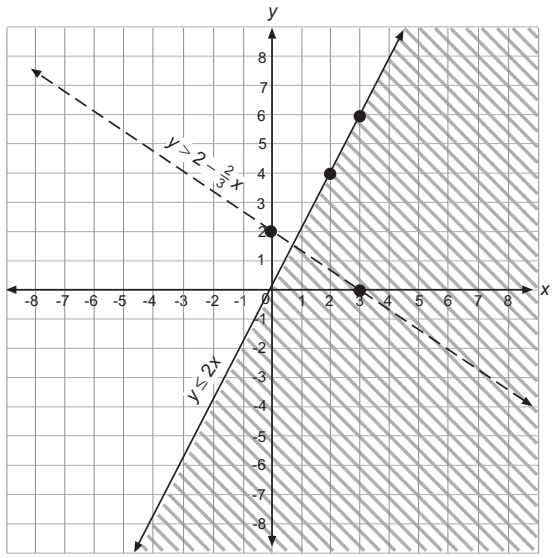
Graph of $2x + 3y > 6$ or $y \leq 2x$ with Inequalities Shaded



Now we change the process to fit the "*or*."

Step 3. Shade the 2nd inequality *below* the solid line using the same shading as in step 2.

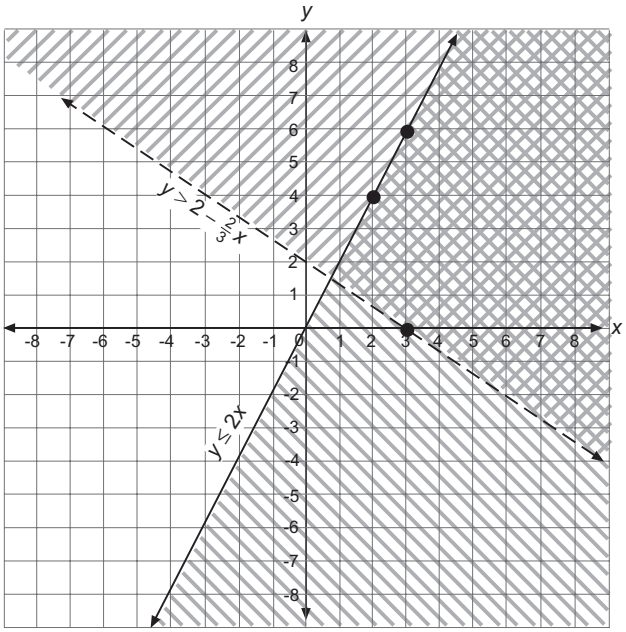
Graph of $2x + 3y > 6$ or $y \leq 2x$ with Inequalities Shaded



Step 4. Because this is now an "or" problem, we want to have as our solution *all* the parts that are shaded. This shows that a solution to *either* inequality is *acceptable*.

Step 5. The solution for $2x + 3y > 6$ **or** $y \leq 2x$ is shown below.

Solution for Graph of $2x + 3y > 6$ or $y \leq 2x$



Now it's your turn to practice.

Practice

Graph the following **inequalities** *on the graphs provided. Refer to pages 500-503 and 512-516 as needed.*

1. $x \geq -2$ **and** $-x + y \geq 1$

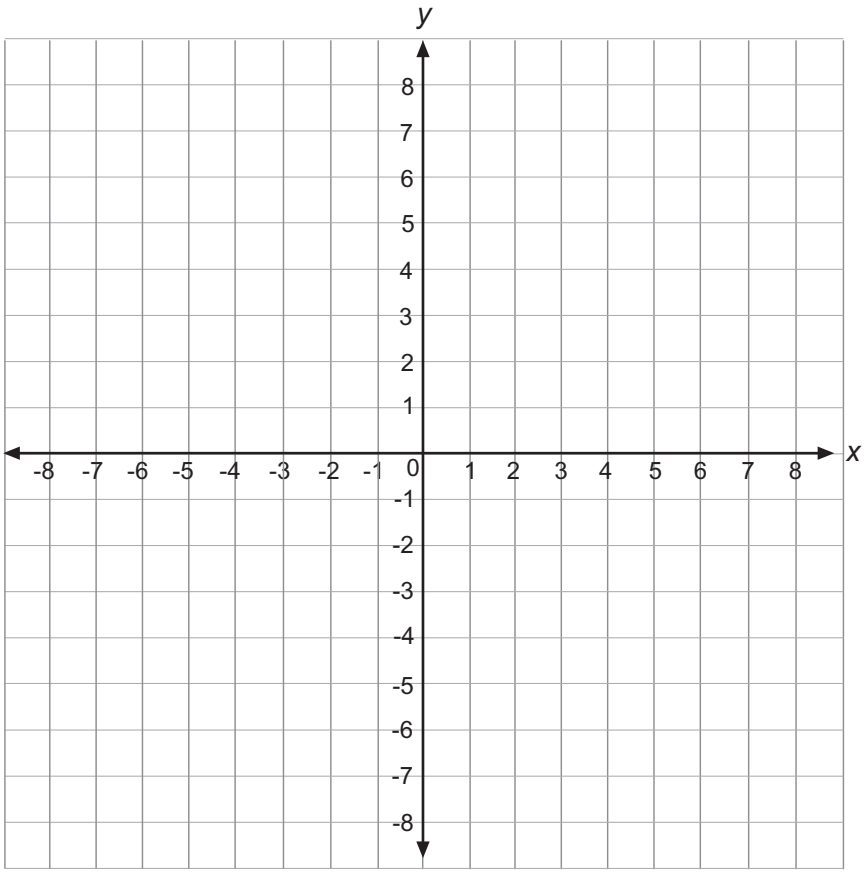
Table of Values

$x \geq -2$	
x	y

Table of Values

$-x + y \geq 1$	
x	y

Graph of $x \geq -2$ and $-x + y \geq 1$



2. $y < -3$ **or** $y \geq 2$

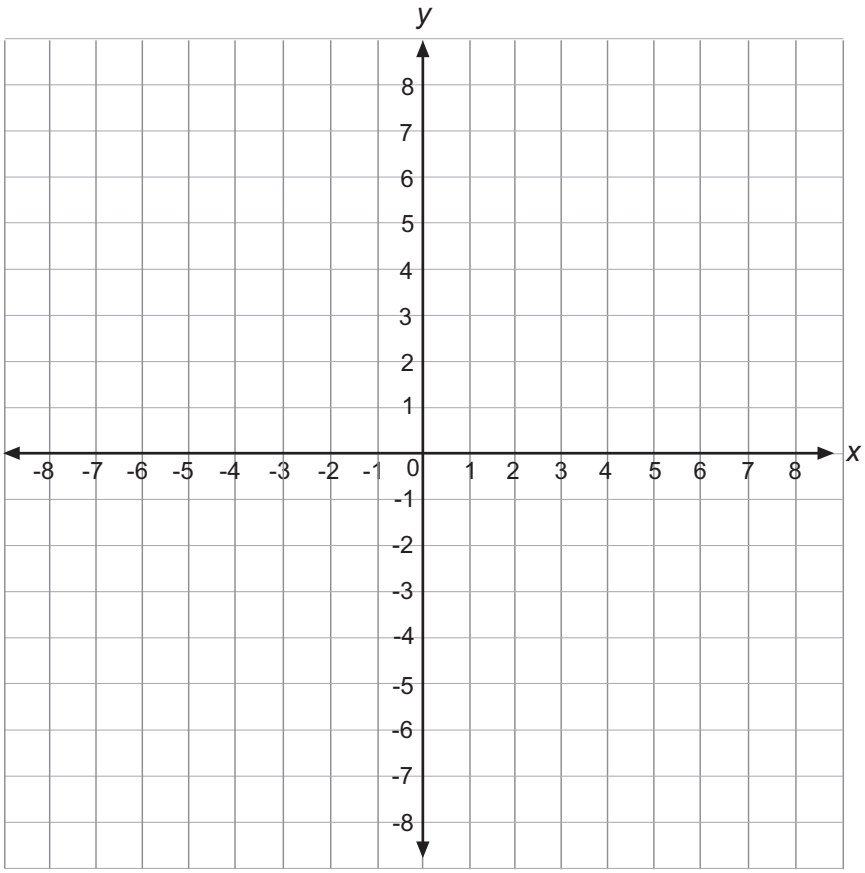
Table of Values

$y < -3$	
x	y

Table of Values

$y \geq 2$	
x	y

Graph of $y < -3$ or $y \geq 2$



3. $x + 2y > 0$ **and** $x - y \leq 5$

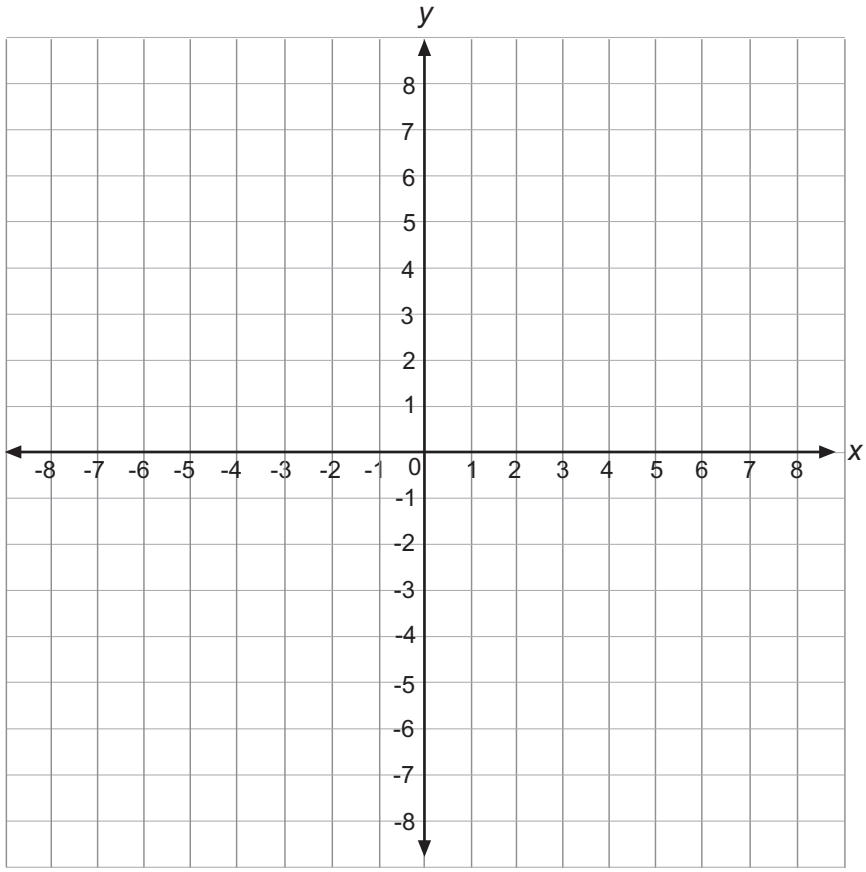
Table of Values

$x + 2y > 0$	
x	y

Table of Values

$x - y \leq 5$	
x	y

Graph of $x + 2y > 0$ and $x - y \leq 5$



4. $x + y > 1$ **or** $x - y > 1$

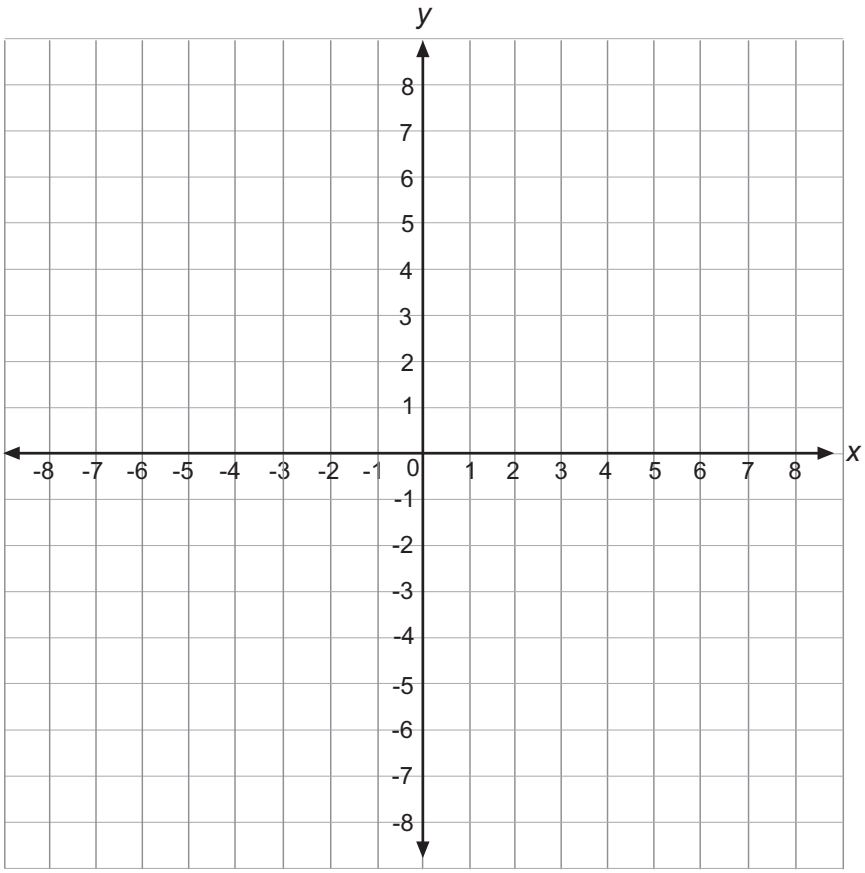
Table of Values

$x + y > 1$	
x	y

Table of Values

$x - y > 1$	
x	y

Graph of $x + y > 1$ or $x - y > 1$



5. $x + y > 1$ **and** $x - y > 1$

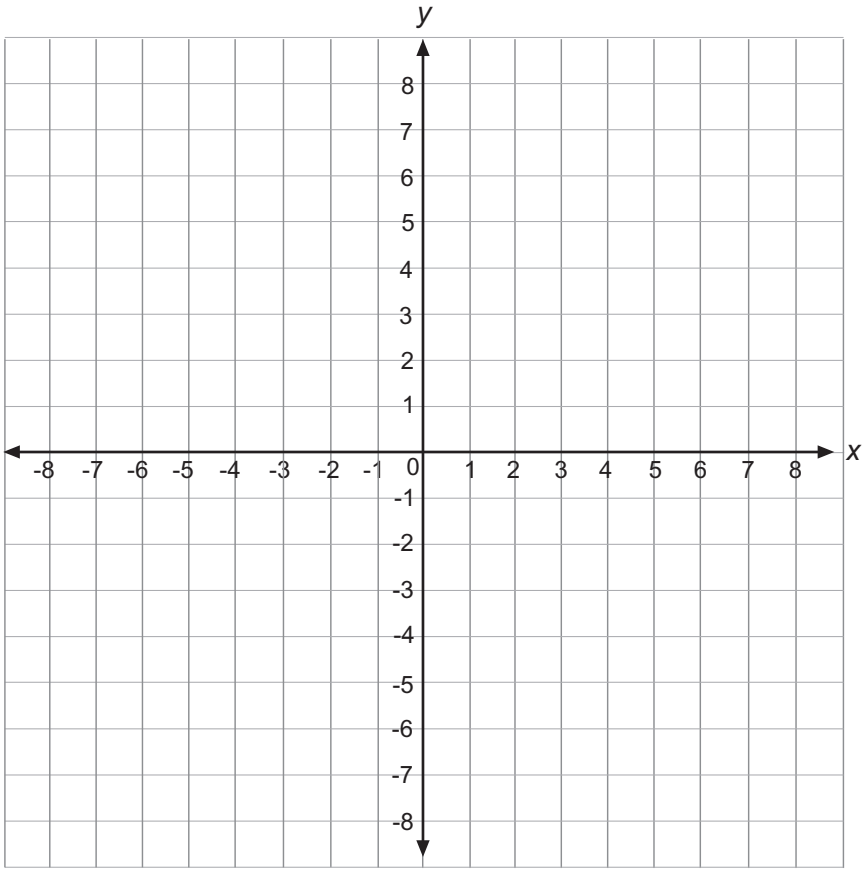
Table of Values

$x + y > 1$	
x	y

Table of Values

$x - y > 1$	
x	y

Graph of $x + y > 1$ and $x - y > 1$



6. $y \leq 3$ **or** $x + y > 4$

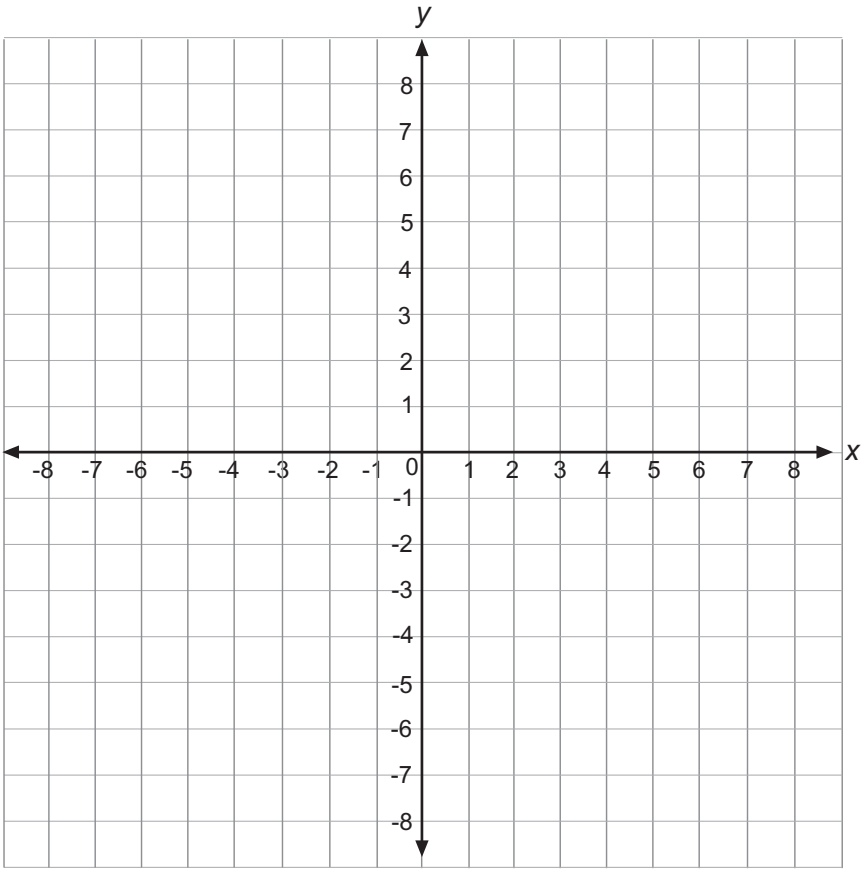
Table of Values

$y \leq 3$	
x	y

Table of Values

$x + y > 4$	
x	y

Graph of $y \leq 3$ and $x + y > 4$



Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. a monomial or sum of monomials; any rational expression with no variable in the denominator

_____ 2. numbers less than zero

_____ 3. a sentence that states one expression is greater than (>), greater than or equal to (≥), less than (<), less than or equal to (≤), or not equal to (≠) another expression

_____ 4. a group of two or more equations that are related to the same situation and share variables

_____ 5. a method used to solve a system of equations in which variables are replaced with known values or algebraic expressions

_____ 6. $ax^2 + bx + c = 0$, where a, b, and c are integers (not multiples of each other) and $a > 0$

_____ 7. expressing a polynomial expression as the product of monomials and polynomials

A. factoring

B. inequality

C. negative numbers

D. polynomial

E. standard form (of a quadratic equation)

F. substitution

G. system of equations

Unit Review

Find the **solution sets**.

1. $(x + 5)(x - 7) = 0$ { _____ , _____ }

2. $(3x - 2)(3x - 6) = 0$ { _____ , _____ }

3. $x(x - 7) = 0$ { _____ , _____ }

4. $x^2 + x = 42$ { _____ , _____ }

5. $x^2 - 10x = -16$ { _____ , _____ }

Solve each of the following. **Show all your work.**

6. Max has a garden 4 feet longer than it is wide. If the area of his garden is 96 square feet, find the dimensions of Max's garden.

   Answer: _____ feet x _____ feet

7. The product of two consecutive positive **even integers** (integers divisible by 2) is 440. Find the integers.

   Answer: _____ and _____

Solve **algebraically**, *then graph each* **system of equations** *on the graphs provided. Refer to pages 471-475, 482-483, 500-503, and 512-516 as needed.*

8. $2x - y = 6$
$x + y = 9$

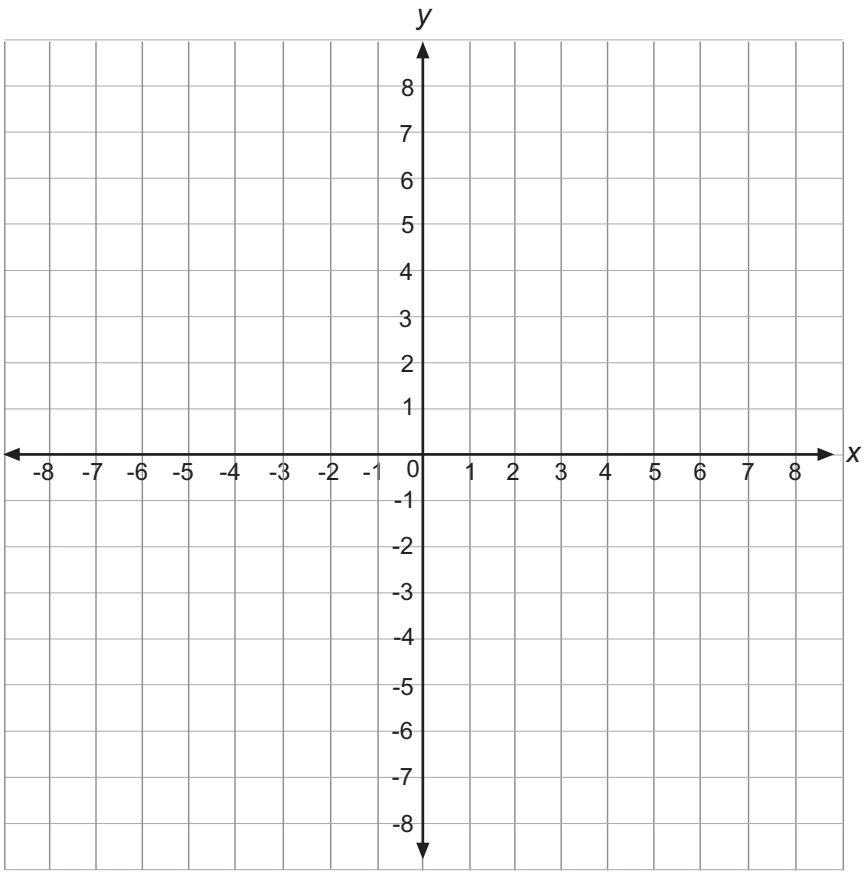
Table of Values

$2x - y = 6$	
x	y

Table of Values

$x + y = 9$	
x	y

Graph of $2x - y = 6$ and $x + y = 9$



9. $x + y = 7$
$3x - 4y = 7$

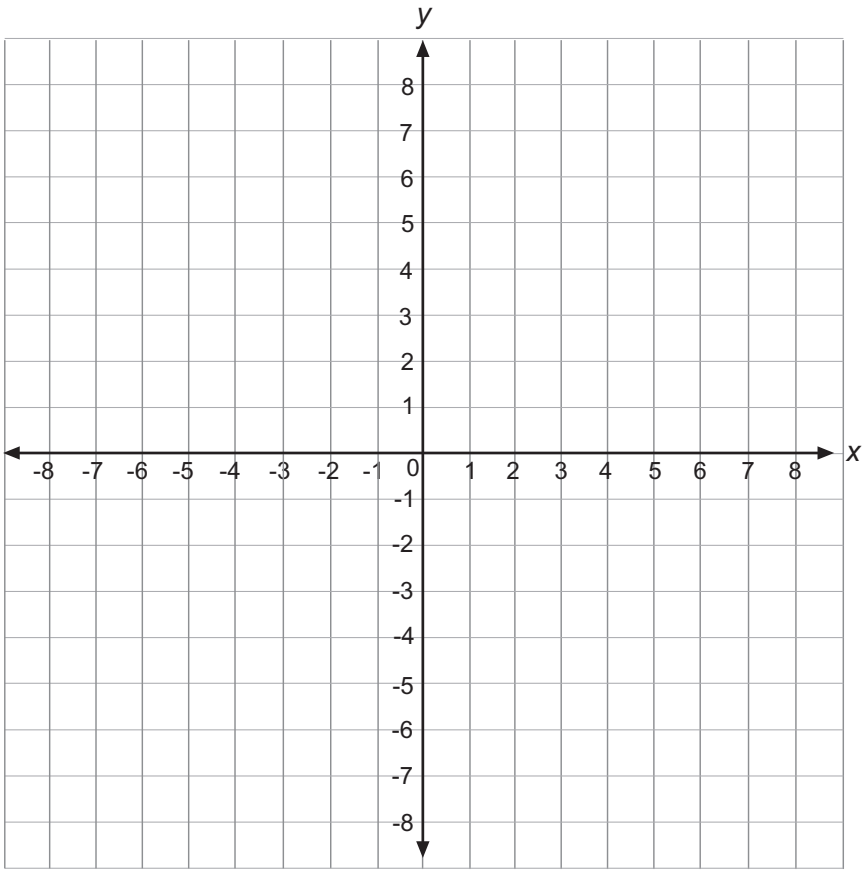
Table of Values

$x + y = 7$	
x	y

Table of Values

$3x - 4y = 7$	
x	y

Graph of $x + y = 7$ and $3x - 4y = 7$



10. $2x - 4y = 8$
$x + 4y = 10$

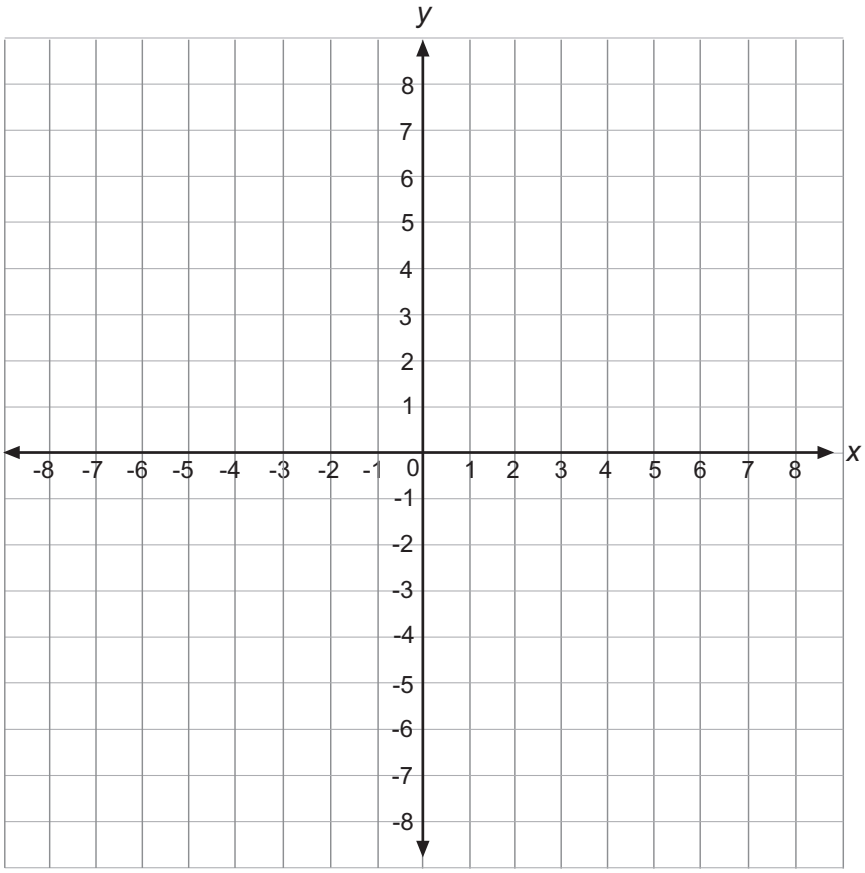
Table of Values

$2x - 4y = 8$	
x	y

Table of Values

$x + 4y = 10$	
x	y

Graph of $2x - 4y = 8$ and $x + 4y = 10$



11. $3x + 2y = 8$
$y = -2$

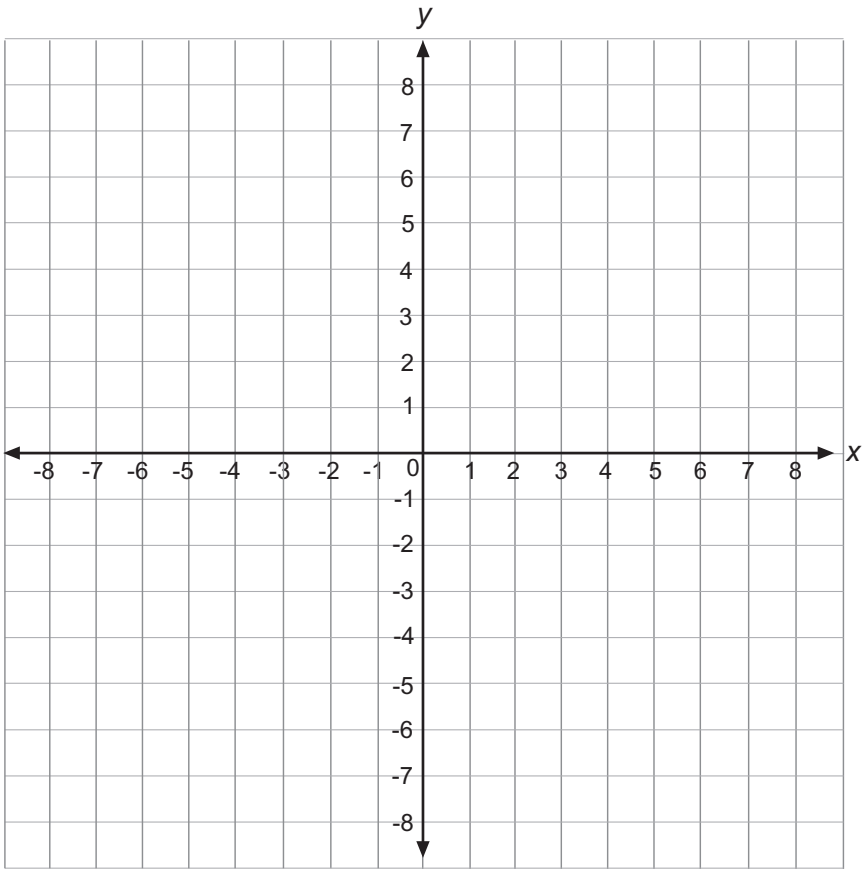
Table of Values

$3x + 2y = 8$	
x	y

Table of Values

$y = -2$	
x	y

Graph of $3x + 2y = 8$ and $y = -2$



12. $3x + 5y = 26$
$2x - 2y = -20$

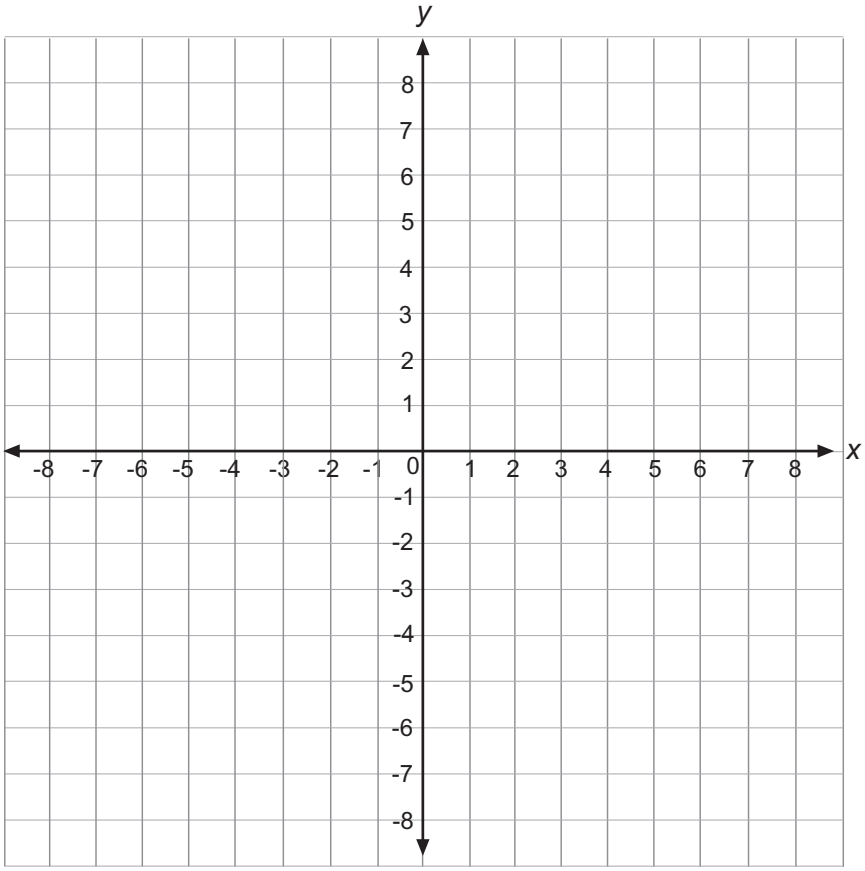
Table of Values

$3x + 5y = 26$	
x	y

Table of Values

$2x - 2y = -20$	
x	y

Graph of $3x + 5y = 26$ and $2x - 2y = -20$



Solve each of the following. **Show all your work.**

13. The sum of two numbers is 52. The larger number is 2 more than 4 times the smaller number. Find the two numbers.

    Answer: _____ and _____

14. The band has 8 more than twice the number of students as the chorus. Together there are 119 students in both programs. How many are in each?

    Answer: chorus = _____ and band = _____

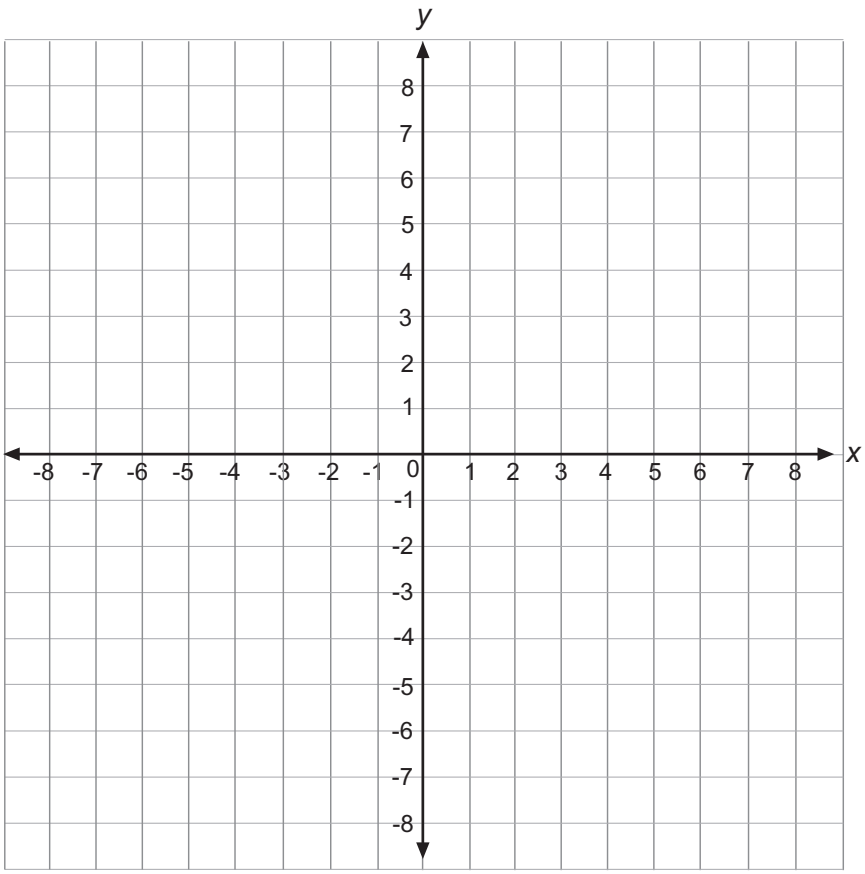
Graph the following **inequalities** *on the graphs provided.*

15. $y > x - 6$

Table of Values

$y > x - 6$	
x	y

Graph of $y > x - 6$

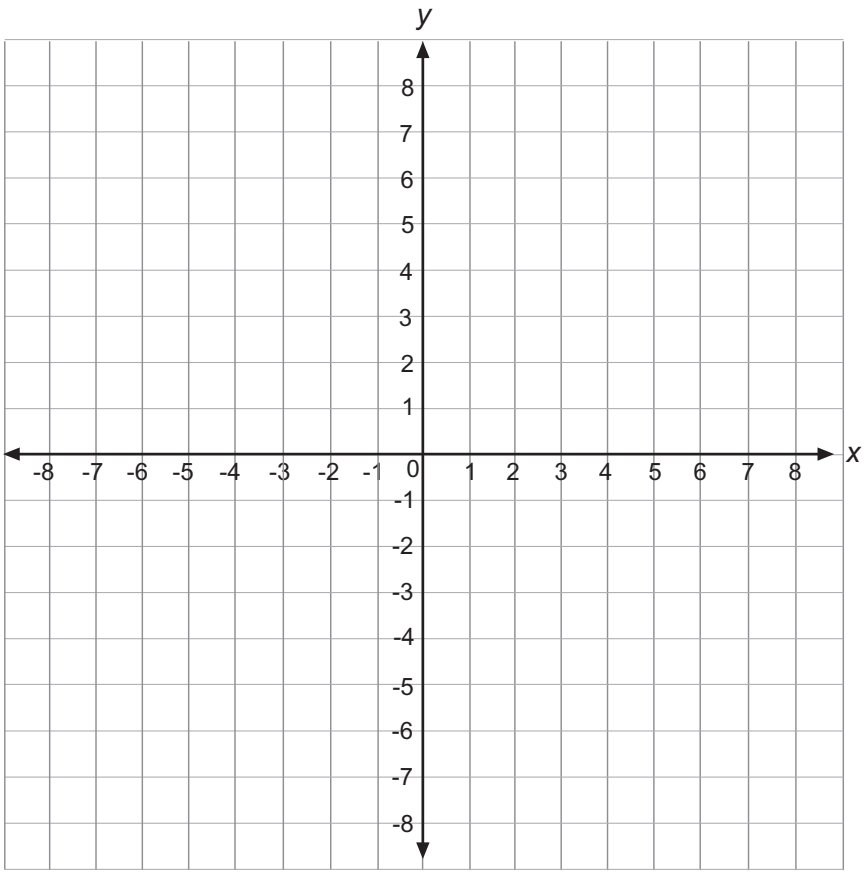


16. $8x - 4y \leq 12$

Table of Values

$8x - 4y \leq 12$	
x	y

Graph of $8x - 4y \leq 12$



17. $y \geq 4x - 3$ **and** $x + y < 0$

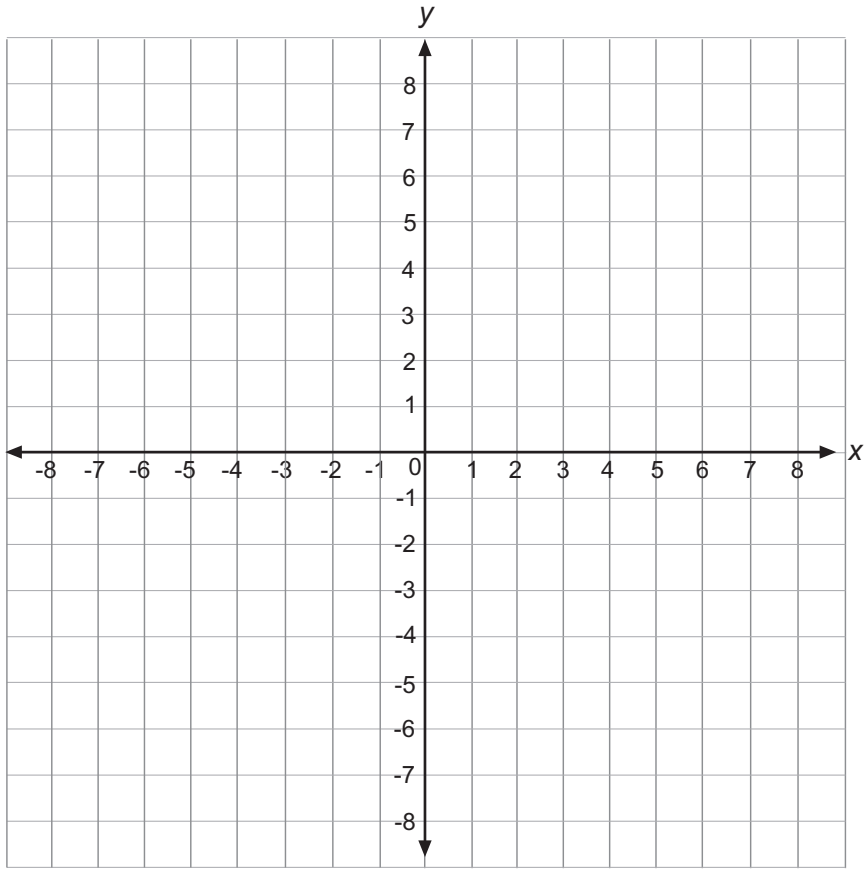
Table of Values

$y \geq 4x - 3$	
x	y

Table of Values

$x + y < 0$	
x	y

Graph of $y \geq 4x - 3$ and $x + y < 0$



18. $x + y > 4$ **or** $y \geq x - 2$

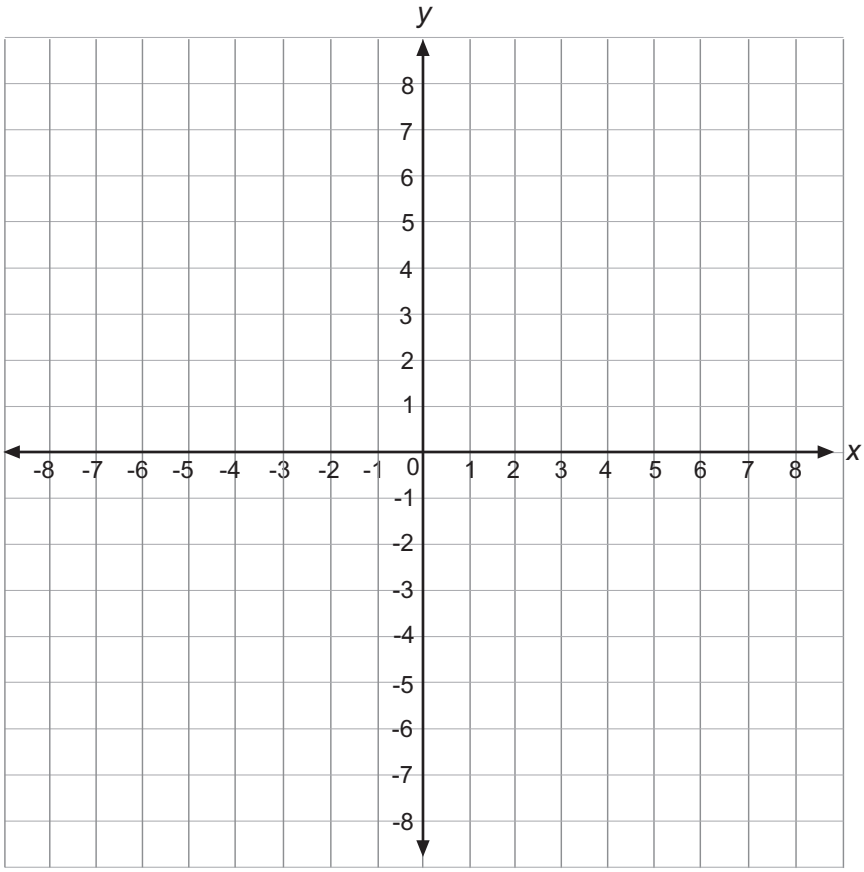
Table of Values

$x + y > 4$	
x	y

Table of Values

$y \geq x - 2$	
x	y

Graph of $x + y > 4$ or $y \geq x - 2$



Unit 9: Sizing Up Geometric Figures

This unit focuses on calculating the perimeter and area of two-dimensional figures and the perimeter, area, and volume of three-dimensional figures.

Unit Focus

Measurement

- Use concrete and graphic models to derive formulas for finding perimeter, area, surface area, circumference, and volume of two- and three-dimensional shapes, including rectangular solids, cylinders, cones, and pyramids. (MA.B.1.4.1)

Algebraic Thinking

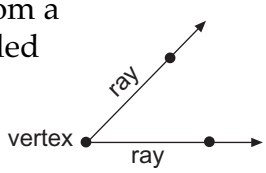
- Determine the impact when changing the parameters of given functions. (MA.D.1.4.2)

Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

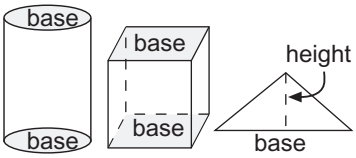
30°-60°-90° triangle ..... a triangle with angles that measure 30°, 60°, and 90°

angle (∠) ..... two rays extending from a common endpoint called the vertex; measures of angles are described in degrees (°)



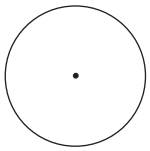
area (*A*) ..... the measure, in square units, of the inside region of a two-dimensional figure
Example: A rectangle with sides of 4 units by 6 units contains 24 square units or has an area of 24 square units.

base (*b*) (geometric) ..... the line or plane of a geometric figure, from which an altitude can be constructed, upon which a figure is thought to rest



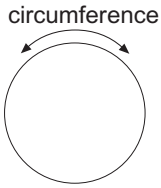
base area ..... the area of a base (*B*) of a geometric solid

center (of a circle) ..... the point from which all points on the circle are the same distance



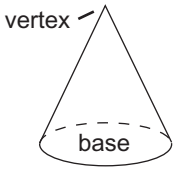
circle ..... the set of all points in a plane that are all the same distance from a given point called the center

circumference (*C*) ..... the distance around a circle



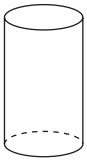
closed figure ..... a two-dimensional figure that divides the plane in which the figure lies into two parts—the part inside the figure and the part outside the figure
Examples: circles, squares, and rectangles

cone ..... a three-dimensional figure with one circular base in which a curved surface connects the base to the vertex

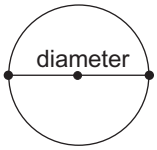


congruent (≅) ..... figures or objects that are the same shape and size

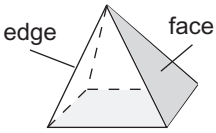
cylinder ..... a three-dimensional figure with two parallel bases that are congruent circles
Example: a can



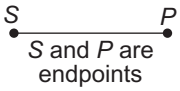
diameter (*d*) ..... a line segment from any point on the circle passing through the center to another point on the circle



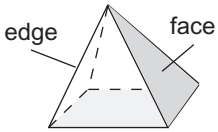
edge ..... the line segment where two faces of a solid figure meet



endpoint ..... either of two points marking the end of a line segment

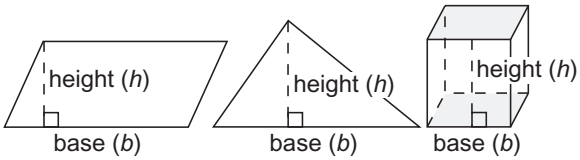


face ..... one of the plane surfaces bounding a three-dimensional figure; a side

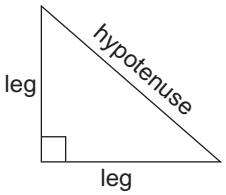


formula ..... a way of expressing a relationship using variables or symbols that represent numbers

height (*h*) ..... a line segment extending from the vertex or *apex* (highest point) of a figure to its base and forming a right angle with the base or plane that contains the base



hypotenuse ..... the longest side of a right triangle; the side opposite the right angle

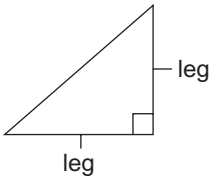


irrational number ..... a real number that cannot be expressed as a ratio of two integers
Example: $\sqrt{2}$

lateral area (*L.A.*) ..... the area of the sides of a geometric solid

lateral face ..... a rectangular side of a rectangular solid

leg ..... in a right triangle, one of the two sides that form the right angle



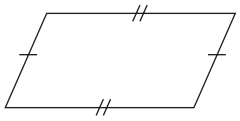
length (*l*) ..... a one-dimensional measure that is the measurable property of line segments

line segment (—) ..... a portion of a line that consists of two defined endpoints and all the points in between
Example: The line segment *AB* is between point *A* and point *B* and includes point *A* and point *B*.

opposite sides ..... sides that are directly across from each other

parallel (||) ..... being an equal distance at every point so as to never intersect

parallelogram ..... a quadrilateral with two pairs of parallel sides



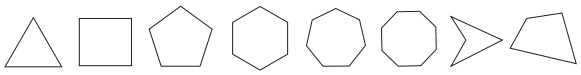
perimeter (*P*) ..... the distance around a polygon

perpendicular (⊥) ..... two lines, two line segments, or two planes that intersect to form a right angle

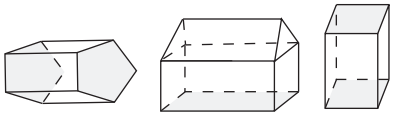
pi (π) ..... the symbol designating the ratio of the circumference of a circle to its diameter; an irrational number with common approximations of either 3.14 or $\frac{22}{7}$

plane figure ..... a two-dimensional figure that lies entirely within a single plane

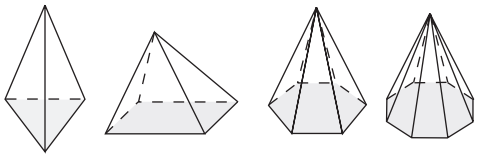
polygon ..... a closed-plane figure, having at least three sides that are line segments and are connected at their endpoints
Example: triangle (3 sides), quadrilateral (4 sides), pentagon (5 sides), hexagon (6 sides), heptagon (7 sides), octagon (8 sides); concave, convex



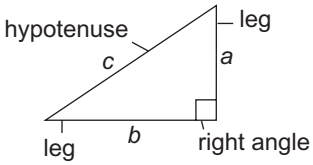
polyhedron ..... a three-dimensional figure in which all surfaces are polygons



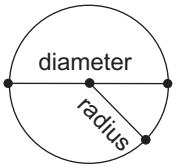
pyramid ..... a three-dimensional figure whose base is a polygon and whose faces are triangles with a common vertex



Pythagorean theorem ..... the square of the hypotenuse (c) of a right triangle is equal to the sum of the square of the legs (a and b), as shown in the equation $c^2 = a^2 + b^2$

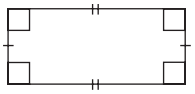


radius (*r*) ..... a line segment extending from the center of a circle or sphere to a point on the circle or sphere; (plural: *radii*)



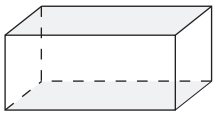
ratio ..... the comparison of two quantities
Example: The ratio of a and b is $a{:}b$ or $\frac{a}{b}$, where $b \neq 0$.

rectangle ..... a parallelogram with four right angles



right angle ..... an angle whose measure is exactly 90°

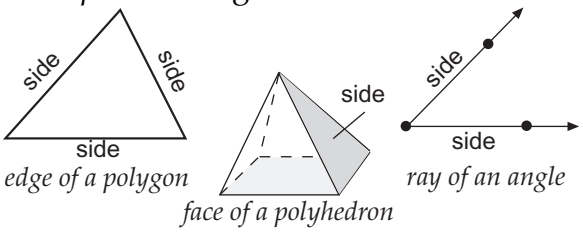
right prism or
rectangular solid ..... a three-dimensional figure (polyhedron) with congruent, polygonal bases and lateral faces that are all parallelograms



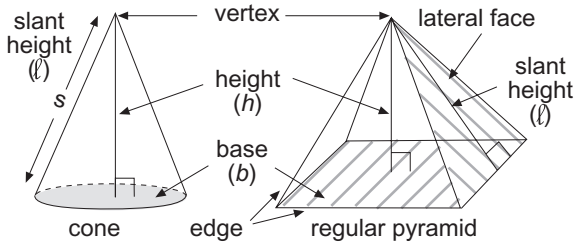
rounded number ..... a number approximated to a specified place
Example: A commonly used rule to round a number is as follows.

- If the digit in the first place after the specified place is 5 or more, *round up* by adding 1 to the digit in the specified place ($\underline{4}61$ rounded to the nearest hundred is 500).
- If the digit in the first place after the specified place is less than 5, *round down* by *not* changing the digit in the specified place ($\underline{4}41$ rounded to the nearest hundred is 400).

side ..... the edge of a polygon, the face of a polyhedron, or one of the rays that make up an angle
Example: A triangle has three sides.

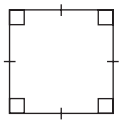


slant height (ℓ) ..... the shortest distance from the vertex of a right circular cone to the edge of its base; the perpendicular distance from the vertex of a pyramid to one edge of its base



solid figures ..... three-dimensional figures that completely enclose a portion of space
Example: rectangular prism, cube, sphere, right circular cylinder, right circular cone, and square pyramid

square ..... a rectangle with four sides the same length



square (of a number) ..... the result when a number is multiplied by itself or used as a factor twice
Example: 25 is the square of 5.

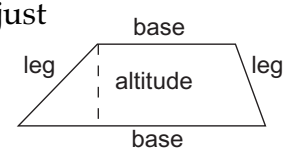
square root ..... a positive real number that can be multiplied by itself to produce a given number
Example: The square root of 144 is 12 or $\sqrt{144} = 12$.

sum ..... the result of adding numbers together
Example: In 6 + 8 = 14,
14 is the sum.

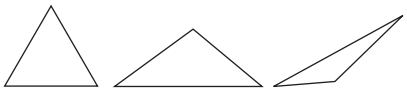
surface area (*S.A.*)
(of a geometric solid) ..... the sum of the areas of the faces and any curved surfaces of the figure that create the geometric solid

three-dimensional
(3-dimensional) ..... existing in three dimensions; having length, width, and height

trapezoid ..... a quadrilateral with just one pair of opposite sides parallel

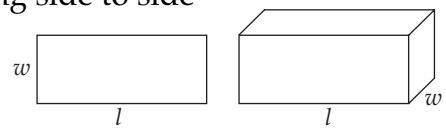


triangle ..... a polygon with three sides



volume (*V*) ..... the amount of space occupied in three dimensions and expressed in cubic units
Example: Both capacity and volume are used to measure empty spaces; however, *capacity* usually refers to *fluid* measures, whereas *volume* is described by cubic units

width (*w*) ..... a one-dimensional measure of something side to side



Unit 9: Sizing Up Geometric Figures

Introduction

Students will practice working with the concepts of perimeter, area, and volume as they relate to different geometric figures. Students will recognize the ways in which formulas are similar, as well as the differences necessary when working with specific figures.

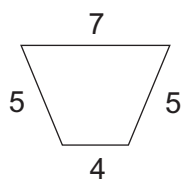
Lesson One Purpose

- Use concrete and graphic models to derive formulas for finding perimeter, area, surface area, circumference, and volume of two- and three-dimensional shapes, including rectangular solids, cylinders, cones, and pyramids. (MA.B.1.4.1)

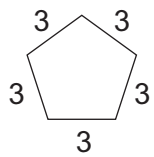
Perimeter of Polygons

Polygons are **closed-plane figures** whose **sides** are **line segments**. Each *side* is connected to two other sides at its **endpoints**. To find the **perimeter (*P*)** of any *polygon*, you add the **lengths (*l*)** of all the sides.

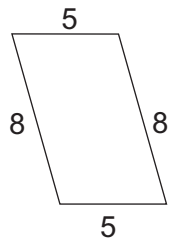
Look at how the *perimeter (P)* was found for each polygon below.



$P = 5 + 7 + 5 + 4$
$= 21$



$P = 3 + 3 + 3 + 3 + 3$
$= 3(5)$
$= 15$



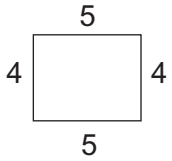
$P = 8 + 5 + 8 + 5$
$= 2(8) + 2(5)$
$= 16 + 10$
$= 26$

Now it's your turn to find the perimeter of each polygon on the following practice.

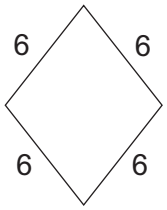
Practice

Find the **perimeter** *of each polygon. Refer to page 547 as needed.*

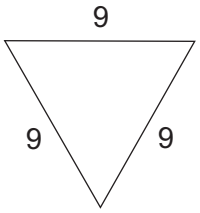
Perimeter is the distance around a polygon.

1. 

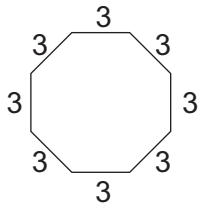
perimeter = _____

2. 

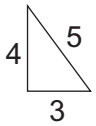
perimeter = _____

3. 

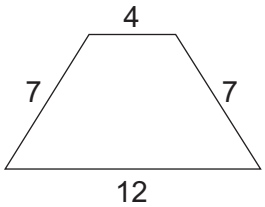
perimeter = _____

4. 

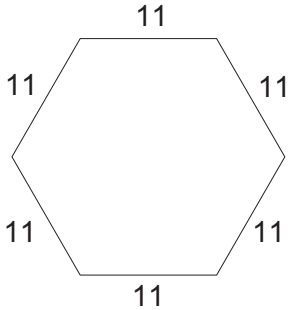
perimeter = _____

5. 

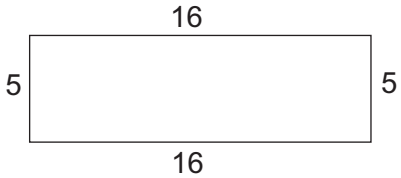
perimeter = _____

6. 

perimeter = _____

7. 

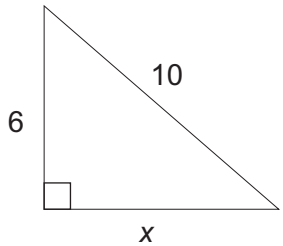
perimeter = _____

8. 

perimeter = _____

Finding the Length of the Missing Side—The Pythagorean Theorem to the Rescue

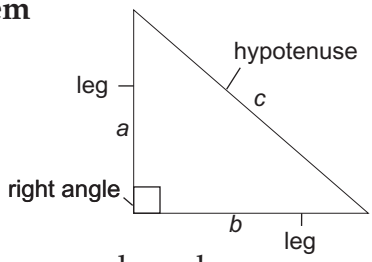
Sometimes, you must figure out the *lengths* (l) of some of the sides before you can begin to find the perimeter. In the first example below, you must use the **Pythagorean theorem** to find the length of the missing side.



Pythagorean theorem

$$a^2 + b^2 = c^2$$

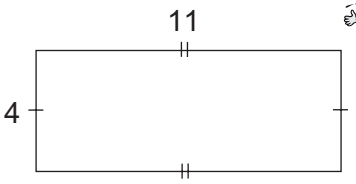
$$\begin{aligned} 6^2 + x^2 &= 102 \\ 36 + x^2 &= 100 \\ x^2 &= 64 \\ x &= 8 \end{aligned}$$



Perimeter = distance around a polygon

$$\begin{aligned} P &= 6 + 10 + 8 \\ &= 24 \end{aligned}$$

In this example, you must use your knowledge of **rectangles** to find some of the missing lengths. Then you can find the perimeter.



rectangle

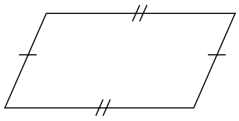
Think:

What do you know to be true of *all* rectangles?

Their **opposite sides** are *always* equal. Therefore,

$$\begin{aligned} P &= 4 + 11 + 4 + 11 \\ &= 2(4) + 2(11) \\ &= 30 \end{aligned}$$

Note: Small slash marks on a figure indicate that the sides so marked are the same length.



- The top is the same length as the bottom.
- The left side is the same length as the right side.

Now try the practice on the following page.

Practice

Find the **perimeter** *of each* **polygon**.

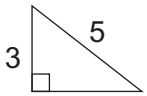
Perimeter is the distance around a polygon.

- For the missing length of a **triangle**, you will need to use the Pythagorean theorem.

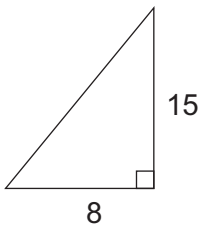
  $a^2 + b^2 = c^2$

- For the missing length of a rectangle, you will need to use your knowledge of rectangles.

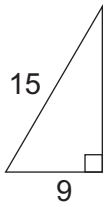
Refer to page 551 as needed.

1. 

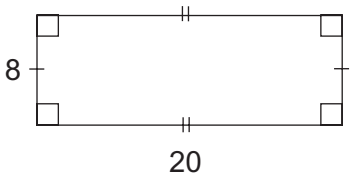
   perimeter = _____

2. 

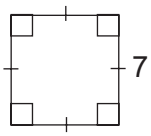
   perimeter = _____

3. 

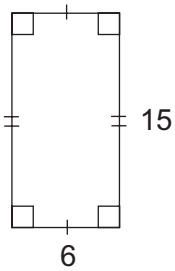
perimeter = _____

4. 

perimeter = _____

5. 

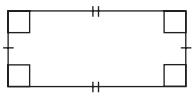
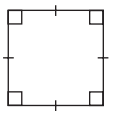
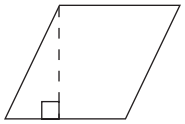
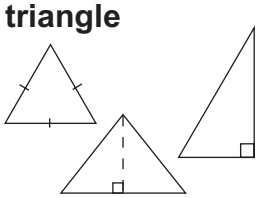
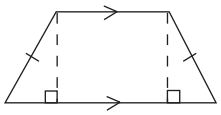
perimeter = _____

6. 

perimeter = _____

Area of Polygons

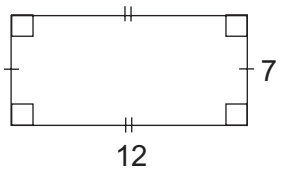
Finding the **area**, or the space inside a figure, requires a bit more work. For each shape, there is a **formula** to use to find its *area*. Let's look at each shape individually.

Polygon	Rule	Formula
rectangle 	Multiply the **base (*b*)** times the **height (*h*)**—some books call it **length (*l*)** times **width (*w*)**. Notice that the base and height are **perpendicular (⊥)** to each other. They form a **right angle**.	$A = b \times h = bh$ or $A = l \times w = lw$
square 	A **square** is a special rectangle. All sides are the same length. You can still use the *formula* $A = bh$, but it may seem easier to use $A = s^2$, with s representing the word *side*.	$A = s^2$
parallelogram 	A **parallelogram** is like a *leaning* rectangle. You use the same formula as for a rectangle, but be sure to notice that the base and height must be *perpendicular line segments.*	$A = bh$
triangle 	A **triangle** is actually half a *parallelogram*. So, we use the *parallelogram* formula and multiply by $\frac{1}{2}$. Be sure to remember that the *base* and *height* must meet at a *right angle*.	$A = \frac{1}{2}bh$
trapezoid 	Notice that a **trapezoid** is like a parallelogram with unequal bases that are both perpendicular to the height. Since choosing one over the other wouldn't seem quite fair, we compromise. We take the average of those two bases and multiply that number times the height. Remember that the bases will always be the two sides that are **parallel (||)** to each other.	$A = \frac{1}{2}h(b_1 + b_2)$

Practice

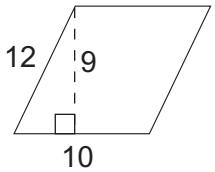
Use the **formulas** *below to find the* **area** *of each* **polygon**. *Refer to pages 551 and 554 as needed.*

1. **rectangle**
   $A = b \times h = bh$
   or
   $A = l \times w = lw$



area = _____

2. **parallelogram**
   $A = bh$



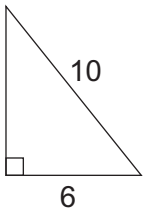
area = _____

3. **triangle**

$A = \frac{1}{2}bh$

Pythagorean theorem

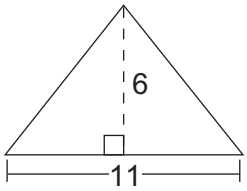
$a^2 + b^2 = c^2$



area = _____

4. **triangle**

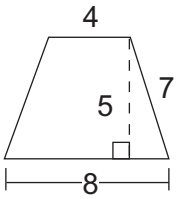
$A = \frac{1}{2}bh$



area = _____

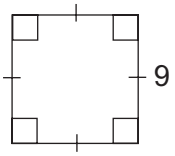
5. **trapezoid**

$A = \frac{1}{2}h(b_1 + b_2)$



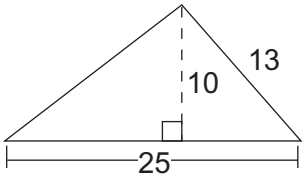
area = _____

6. **square**
$A = s^2$



area = _____

7. **triangle**
$A = \frac{1}{2}bh$



area = _____

8.

trapezoid

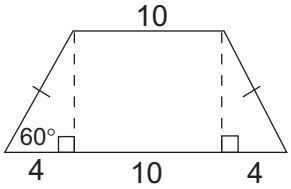
$A = \frac{1}{2}h(b_1 + b_2)$

or

30°-60°90° triangle

Pythagorean theorem

$a^2 + b^2 = c^2$



area = _____

Remember: 30°-60°-90° triangle

- the short **leg** is always opposite the 30° **angle** and is $\frac{1}{2}$ the length of the **hypotenuse**
- the long *leg* is always opposite the 60° *angle* and is $\sqrt{3}$ times the length of the short leg

  or
- the length of the short leg can be found by dividing the length of the long leg by $\sqrt{3}$

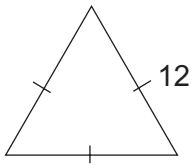
9.

triangle

$A = \frac{1}{2}bh$

Pythagorean theorem

$a^2 + b^2 = c^2$



area = _____

10.

rectangle

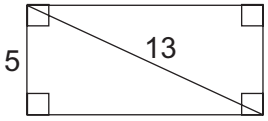
$A = b \times h = bh$

or

$A = l \times w = lw$

Pythagorean theorem

$a^2 + b^2 = c^2$



area = _____

Answer the following.

11. Double the lengths of the triangle in number 9. What is the area now? How does this compare to the original area?

_____

12. Double the lengths of the rectangle in number 10. Find the area. How does this area compare with the original area?

_____

13. Write a rule suggesting what happens to area when dimensions are

doubled. _____

_____

What about tripled? _____

_____

Or halved? _____

_____

Practice

Write **True** *if the statement is correct. Write* **False** *if the statement is not correct.*

_____ 1. A closed-plane figure, having at least three sides that are line segments and are connected at their endpoints, is called a *polygon*.

_____ 2. To solve for the *base*, you need to find the distance around a polygon.

_____ 3. *Area* is the measure, in square units, of the inside region of a two-dimensional figure.

_____ 4. A *rectangle* is a polygon with three sides.

_____ 5. A *square* is a rectangle with four sides the same length.

_____ 6. When you have a quadrilateral with two pairs of parallel sides, its called a *trapezoid*.

_____ 7. A parallelogram with four right angles is a *triangle.*

_____ 8. A *parallelogram* is a quadrilateral with just one pair of opposite sides that are parallel.

_____ 9. The line or plane of a geometric figure (from which an altitude can be constructed and upon which a figure is thought to rest) is called the *perimeter*.

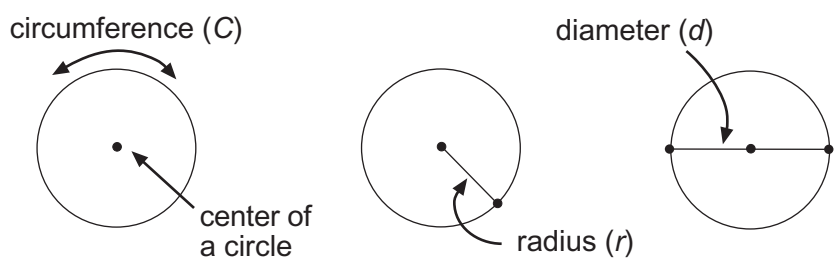
_____ 10. When you have two lines, two line segments, or two planes that intersect to form a right angle, they are *perpendicular*.

Lesson Two Purpose

- Use concrete and graphic models to derive formulas for finding perimeter, area, surface area, circumference, and volume of two- and three-dimensional shapes, including rectangular solids, cylinders, cones, and pyramids. (MA.B.1.4.1)
- Determine the impact when changing the parameters of given functions. (MA.D.1.4.2)

Circles

Circles are special in that they have their own formulas for perimeter and area. In fact, *circles* are so special that they have a different term for perimeter. We say **circumference (*C*)** when we are referring to the *perimeter of a circle*, or the distance around it. The only other dimension we need to find for either the area or *circumference* of a circle is the **radius (*r*)**. Remember that the *radius* is the distance from the **center of the circle** to any point on the circle. Also, when we know the **diameter (*d*)** of the circle, which is the distance across the *center of the circle*, we can divide it in half to find the length of the radius.



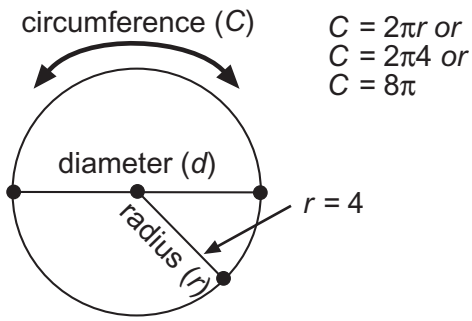
circles

Another very important piece of information we use with circles is the special number π. We use this Greek letter (called **pi**) to represent the number which is the **ratio** of the circumference of a circle to its *diameter*. π is actually an **irrational number** and is approximately equal to ($\approx$) $\frac{22}{7}$, or 3.14. So, if you are asked for an approximation or a *rounded* value involving pi, you use one of the numbers above. But if you are asked for an *exact* value, use the symbol π in your answer.

Now, let's review the formulas for circumference and area of circles.

The formula for circumference is $C = 2\pi r$.

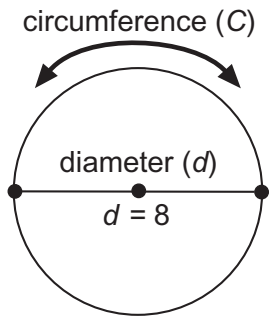
If $r = 4$,
$C = 2 \times \pi \times 4$ *or* 8π.



$C = 2\pi r$ *or*
$C = 2\pi 4$ *or*
$C = 8\pi$

The formula for circumference is also $C = \pi d$.

If $d = 8$,
$C = \pi \times 8$ *or* 8π.

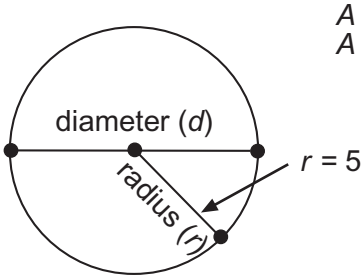


$C = \pi d$ *or*
$C = 8\pi$

The formula for area is $A = \pi r^2$.

If $r = 5$,
$A = \pi 5^2$ *or* 25π.

$A = \pi r^2$ *or*
$A = \pi 5^2$ *or*
$A = 25\pi$



I know you remember these now, so let's practice a few on the next page.

Practice

Use the **formulas** *below to find the* ***exact*** **circumference** *and* **area** *for each* **circle** *with the given information. Refer to page 562 as needed.*

Remember: *Exact* means to use the π symbol in your answer.

circumference	**circumference**	**area**
$C = 2\pi r$	$C = \pi d$	$A = \pi r^2$

1. $r = 4$

   circumference = _____

   area = _____

2. $r = 3$

   circumference = _____

   area = _____

3. $r = 5$

   circumference = _____

   area = _____

4. $r = 6$

circumference = _____

area = _____

5. $r = 7$

circumference = _____

area = _____

6. $r = 10$

circumference = _____

area = _____

Answer the following.

7. Compare the circumferences you found for number 2 and number 4. Compare the circumferences you found for number 3 and number 6. Suggest a rule for what effect doubling the radius of a circle has on the circumference.

_____

_____

_____

8. Compare the areas you found for number 2 and number 4. Compare the areas you found for number 3 and number 6. Suggest a rule for the effect doubling the radius of a circle has on the area of the circle.

_____

_____

_____

9. Repeat number 7 and number 8, but triple the dimensions. What happens?

_____

_____

_____

Suggest a rule for your results.

_____

_____

_____

Practice

Use the **formulas** *below to find* **circumference** *and* **area** *with the given information for each* **circle**. **Round answers to the nearest hundredth.** *Refer to page 562 as needed.*

Remember: π is actually an *irrational number* with an approximate value of $\frac{22}{7}$ or 3.14.

circumference	**circumference**	**area**
$C = 2\pi r$	$C = \pi d$	$A = \pi r^2$

1. $r = 25$

   circumference = _____

   area = _____

2. $r = 8$

   circumference = _____

   area = _____

3. $d = 16$

   circumference = _____

   area = _____

4. $d = 22$

   circumference = _____

   area = _____

5. Did you get the same answer for number 2 and number 3? _____

   Why? _____

   _____

Use your **algebra skills** *to find the* **radius** *when given the* **areas** *or* **circumferences**.

First check out the following examples.

Example 1

$$\begin{aligned} C &= 60\pi && \longleftarrow \text{Fill in the formula for } \textbf{circumference}. \\ 2\pi r &= 60\pi && \longleftarrow \text{Divide both sides by 2.} \\ \pi r &= 30\pi && \longleftarrow \text{Divide both sides by } \pi. \\ \text{So,} \quad r &= 30 \end{aligned}$$

Example 2

$$\begin{aligned} A &= 49\pi && \longleftarrow \text{Fill in the formula for } \textbf{area}. \\ \pi r^2 &= 49\pi && \longleftarrow \text{Divide both sides by } \pi. \\ r^2 &= 49 && \longleftarrow \text{Take the } \textbf{square root} \text{ of both sides.} \\ \text{So,} \quad r &= 7 \end{aligned}$$

Find the **radius** *for each of the following* **circles** *with the given information.*

6. $C = 24\pi$

   radius = _____

7. $C = 16\pi$

   radius = _____

8. $C = 50\pi$

   radius = _____

9. $C = 18\pi$

   radius = _____

10. $C = 15\pi$

    radius = _____

11. $A = 9\pi$

radius = _____

12. $A = 50\pi$

radius = _____

13. $A = 4\pi$

radius = _____

14. $A = 24\pi$

radius = _____

15. $A = 11\pi$

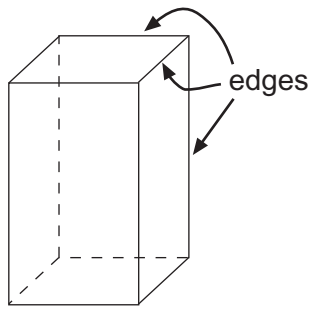
radius = _____

Lesson Three Purpose

- Use concrete and graphic models to derive formulas for finding perimeter, area, surface area, circumference, and volume of two- and three-dimensional shapes, including rectangular solids, cylinders, cones, and pyramids. (MA.B.1.4.1)
- Determine the impact when changing the parameters of given functions. (MA.D.1.4.2)

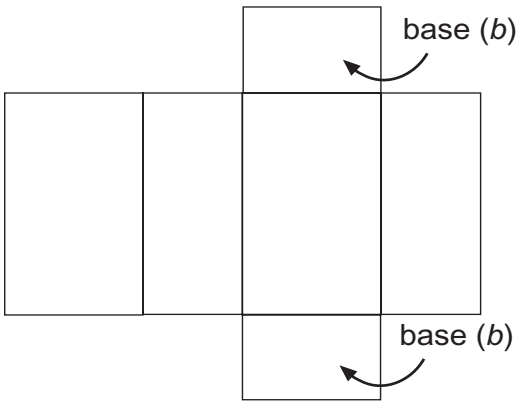
Areas of Rectangular Solids

A rectangular solid is a **three-dimensional** figure or **polyhedron**. A **right prism** or **rectangular solid** has **congruent (≅)**, polygonal bases (*b*) and **lateral faces**, or sides that are all parallelograms. Two of the sides are *bases* (usually the bottom and the top) and the other sides are called *lateral faces*.



rectangular solid

If we cut around some of the **edges**, unfold the **solid figure**, and then flatten it out, it would look like this.



rectangular solid flattened out

Lateral Areas of Rectangular Solids

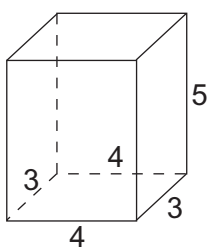
You now see that the lateral faces are all connected and form a large rectangle with two bases attached at the top and bottom. We can find the **lateral area (*L.A.*)**, the area of the lateral faces, just as we would for any other rectangle, using the formula base times (x) height. Notice that the base of the big rectangle is the same as the perimeter of the base of the right prism or rectangular solid we started out with. Also, the height of that big rectangle is the same as the height of the original solid. With this in mind, we can see that the lateral area of the solid can be found by multiplying the perimeter of a base times (x) the height of the solid.

So, *lateral area* = (*L.A.*) = perimeter (*P*) of the base times (x) the height (*h*) of a base.

lateral area
$L.A. = Ph$

Remember: *Opposite sides* of a rectangle are the same length.

Let's look at an example of a rectangular solid.



rectangular solid

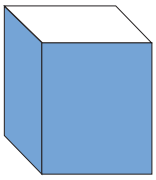
Find the perimeter of a base.
$4 + 3 + 4 + 3 = 14$

Find the height of the solid.
5

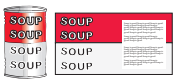
$L.A. = Ph = 14 \times 5 = 70$

$L.A. = 70$

You might think of finding lateral area (*L.A.*) of a rectangular solid as finding *exactly* how much paper it would take to cover *just* the sides (not the top and bottom) of a rectangular package (like *just* the sides of a soup can).



paper needed to cover just *the sides of a package*



paper needed to cover just *the sides of a can*

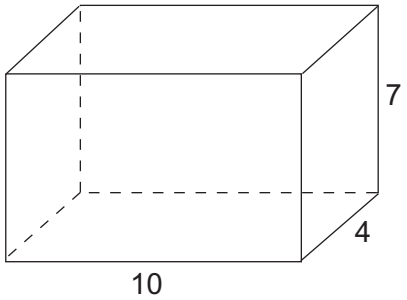
Now, you try with the practice on the following page.

Practice

Use the formula below to find the **lateral area** *of each* **rectangular solid**.

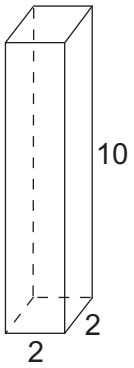
lateral area
L.A. = Ph

1.



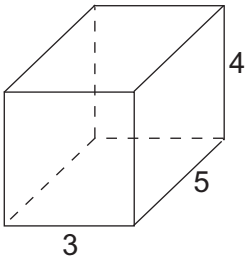
lateral area = _____

3.



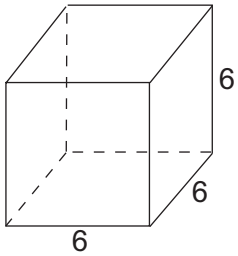
lateral area = _____

2.



lateral area = _____

4.



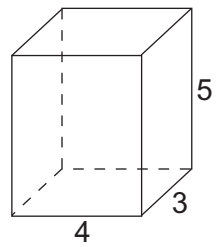
lateral area = _____

Note: Please have your teacher check these answers. You will need them to do the next practice.

Surface Area of Rectangular Solids

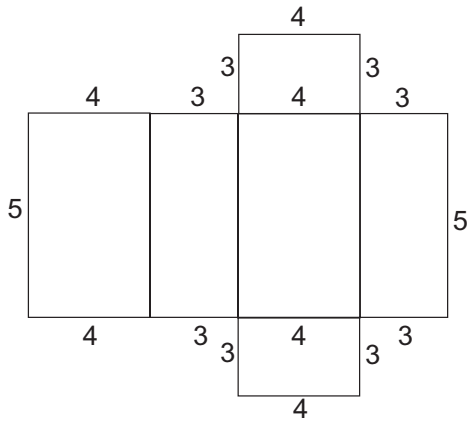
We are now ready to find the **surface areas (*S.A.*)** of rectangular solids. The *surface area* (*S.A.*) is the **sum** of the areas of each **face** of the rectangular solid. Surface area is the lateral area (*L.A.*), *plus* the area of the two bases.

This is our *original* rectangular solid.



rectangular solid

This is how the rectangular solid would look *flattened out*.



rectangular solid flattened

Remember that the $L.A. = 70$. Now, find the **base area** or the area of a base (B) of a solid figure. Since the base is a rectangle, we multiply the length (l) of the base (B) times the height (h) of the base (B).

area of the base
$B = lh$

So, for the area of a base

$B = l \times h =$
$B = 3 \times 4 = 12$

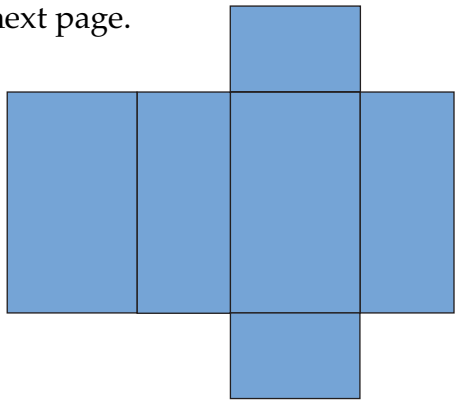
Now for the finishing touch.

surface area of a rectangular solid
$S.A. = L.A. + 2B$

surface area	=	lateral area	+	2(*base area*)	=	
$S.A.$	=	$L.A.$	+	$2B$		
$S.A.$	=	70	+	2(12)	= 94	

Unlike the illustration on page 571, where the rectangular solid was imagined to be covered like a *soup can*, *imagine* that the rectangular solid needs to be covered on all six sides with paper, like a present. However, there are *no* places where the covering overlaps!

It's your turn, again, to practice on the next page.



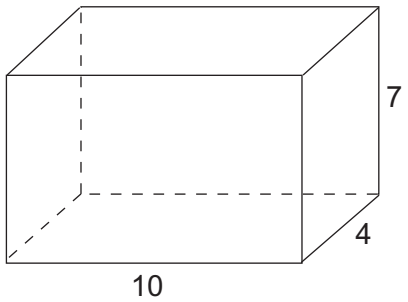
paper needed to cover all six sides of package

Practice

Use the **formula** *below to find the* **surface area** *for each* **rectangular solid** *from numbers 1-4 in the* **previous practice**. *The figures have been recopied below. Refer to pages 570-574 as needed.*

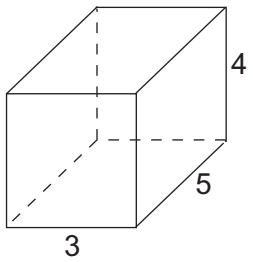
surface area of a rectangular solid
$S.A. = L.A. + 2B$

1.



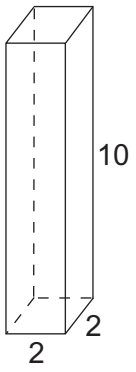
surface area = _____

2.



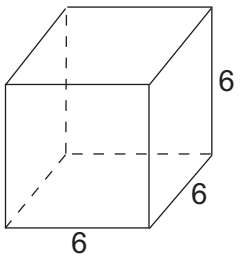
surface area = _____

3.



surface area = _____

4.



surface area = _____

Answer the following.

5. Double each dimension for number 3 and find the surface area.
   Example: 2 x 2 = 4.

   surface area = _____

6. Triple each dimension for number 3 and find the surface area.
   Example: 2 x 3 = 6.

   surface area = _____

7. Suggest a rule about the effect that expanding the dimensions of a figure has on the figure's total area.

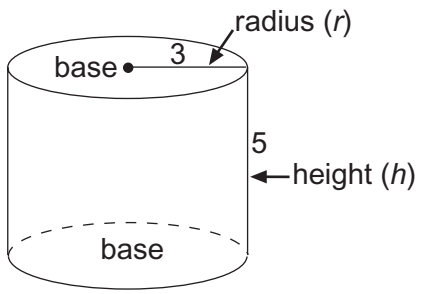
   _____

   _____

   _____

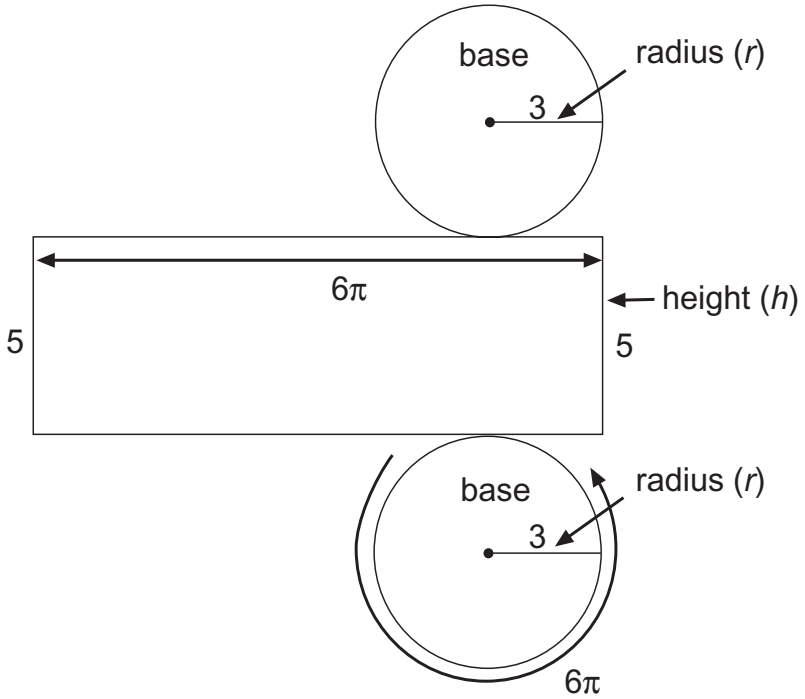
Areas of Cylinders

Cylinders are much like rectangular solids except that they have circular bases and only one lateral face. Look at the *cylinder* below.



cylinder

If we slice around the bases and straight up the face, we get a figure that looks like the following.



cylinder flattened out

Lateral Area of Cylinders

The lateral face of a cylinder is a rectangle whose length is the same as the circumference (C) of a circle, because the cylinder was formed by wrapping the rectangle around the circles. We can find the lateral area ($L.A.$) of the cylinder in much the same way as we did with the rectangular solids. Lateral area is still perimeter (P) of a base times (x) the height (h) of the cylinder. The only difference is that now the perimeter is actually the circumference of the circle.

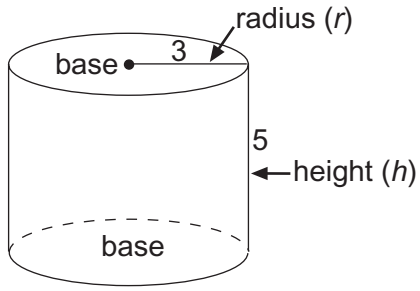
Remember: Circumference = $2\pi r$

So, $L.A. = Ph = 2\pi rh$.

lateral area of a cylinder
$L.A. = 2\pi rh$

The lateral area of the cylinder below (recopied from the previous page) is

$$L.A. = 2\pi rh$$
$$2 \times \pi \times 3 \times 5 = 30\pi$$



cylinder

Note: The symbol π (pi) is used in the answer to state the *exact* lateral area of a cylinder.

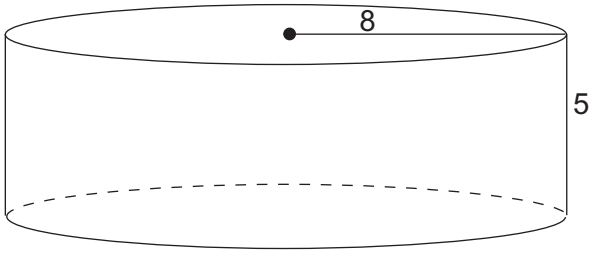
Using the previous example, try the following practice.

Practice

Use the **formula** *below to find the* **lateral area (*L.A.*)** *for each* **cylinder**. *Use the symbol* π *to state the* ***exact*** **lateral area** *of the* **cylinder** *in your answer. Refer to pages 578 and 579 as needed.*

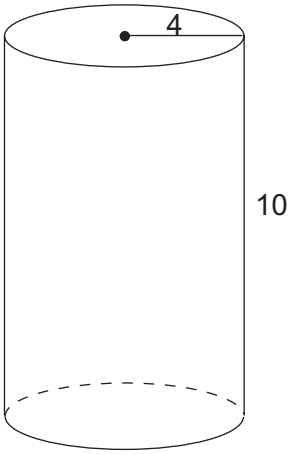
lateral area of a cylinder
$L.A. = 2\pi rh$

1.

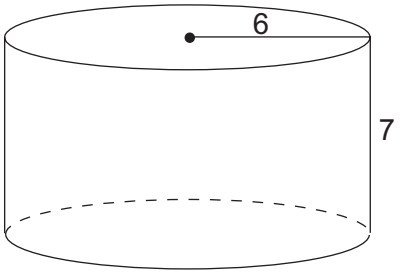


lateral area = _____

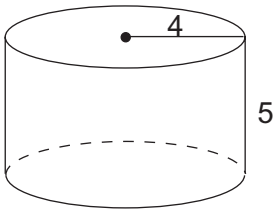
2.



lateral area = _____

3. 

lateral area = _____

4. 

lateral area = _____

Surface Area of Cylinders

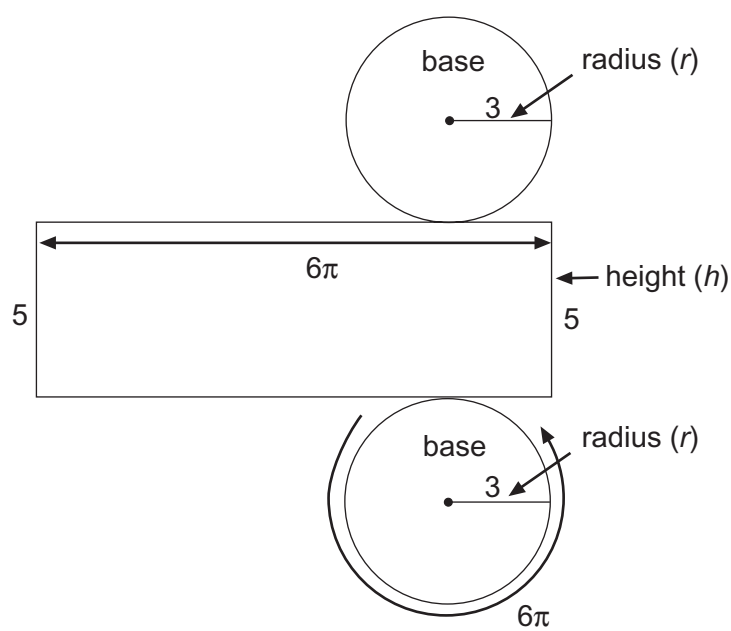
Just as we did with rectangular solids, we simply add the areas of the bases to the lateral area to find the surface area. Remember, the bases are circles, so we must use the formula for the area of a circle. For area of a circle, area equals pi (π) times (x) the radius (r) squared (r^2).

Remember: The **square of a number** results when a number is multiplied by itself.

area of a circle
$A = \pi r^2$

Our generic formula $S.A. = L.A. + 2B$ can be adjusted using formulas for circles and becomes $S.A. = 2\pi rh + 2\pi r^2$. It is easier to remember the generic formula and just adjust it to fit the shapes involved.

surface area of a cylinder
$S.A. = L.A. + 2B$
or
$S.A. = 2\pi rh + 2\pi r^2$



cylinder flattened out

So, $S.A.$ for the figure above is $L.A. + 2B = 30\pi + 18\pi = 48\pi$.

Guess what? It's your turn again to practice.

Practice

Use either **formula** *below to find the* **surface area** *for each* **cylinder** *from the numbers 1-4 in the* **previous practice**. *The figures have been recopied below.*

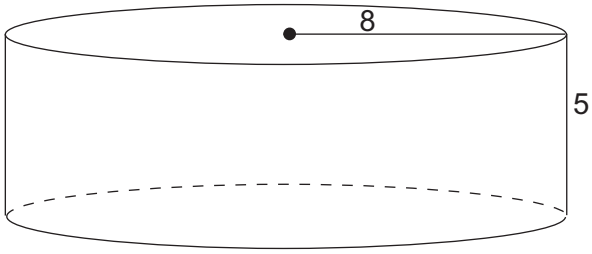
surface area of a cylinder

$S.A. = 2\pi rh + 2\pi r^2$

or

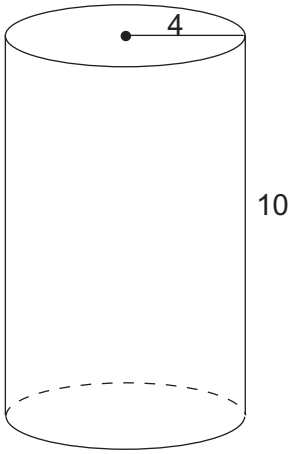
$S.A. = L.A. + 2B$

1.

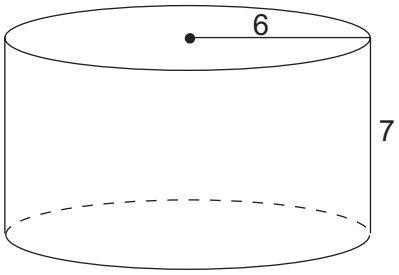


surface area = _____

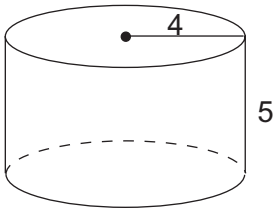
2.



surface area = _____

3. 

surface area = _____

4. 

surface area = _____

Answer the following.

5. Double each dimension for number 3 and find the surface area.
   Example: 6 x 2 = 12.

   surface area = _____

6. Triple each dimensions for number 3 and find the surface area.
   Example: 6 x 3 = 18.

   surface area = _____

7. Suggest a rule about the effect that expanding the dimensions of a figure has on the figure's surface area.

   _____

   _____

   _____

8. Compare the dimensions and surface area for number 1 and 2, then number 2 and 4. Are the surface areas related in the ways you thought they would be? Explain.

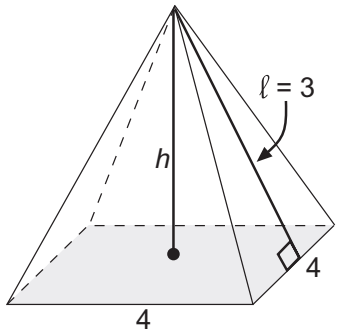
   _____

   _____

   _____

Areas of Pyramids

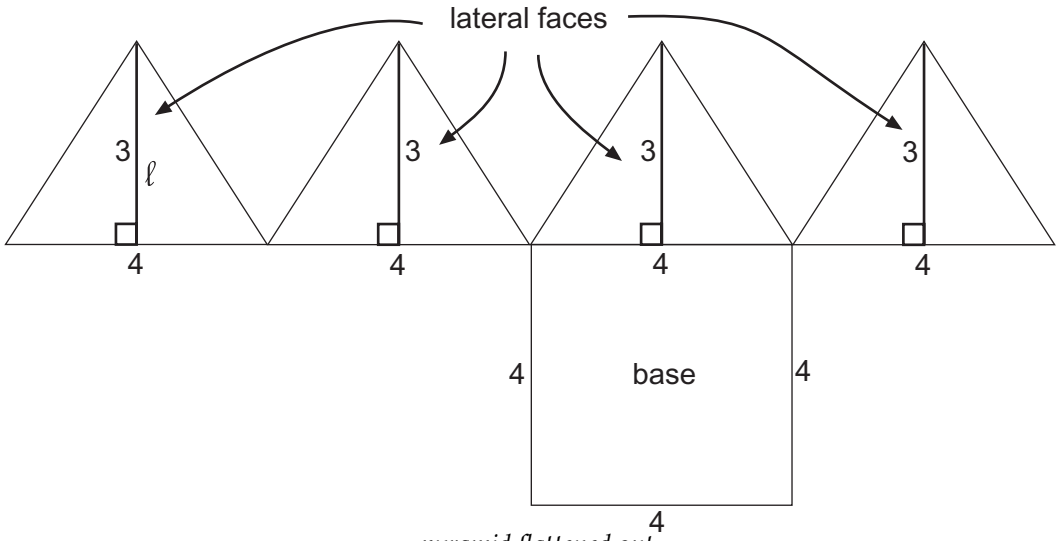
Believe it or not, **pyramids** are somewhat like rectangular solids. They have lateral faces that are triangles. *Pyramids* have bases that are polygons, but each pyramid has only one base.



pyramid

Lateral Areas of Pyramids

If we cut along some of the edges of the pyramid above and flatten it out, it we get a figure that looks like this.



pyramid flattened out

You can see that all the lateral faces are triangles. They wrap around the base and then lean in to touch each other at the top of the pyramid. Because the lateral faces are triangles, and the area of a triangle is half the area of a rectangle, we remember to use the formula for area of a triangle to find the area of each face.

In a triangle, the area of the triangle equals one half the base times (x) the height.

area of a triangle

$$A = \frac{1}{2}bh$$

Remember: The base and the height must be perpendicular to each other.

You must use the height of the triangle, *not* the pyramid, to find the area of a face. To help avoid confusion, we call the height of a triangular face ℓ. This (ℓ) stands for **slant height** because the triangles must *lean in* to form the pyramid. The *height* of the pyramid, *straight up the middle*, is labeled h.

For a rectangular solid, $L.A. = Ph$. So, in a triangle, $L.A. = \frac{1}{2}P\ell$. Do you see the similarities in the formulas?

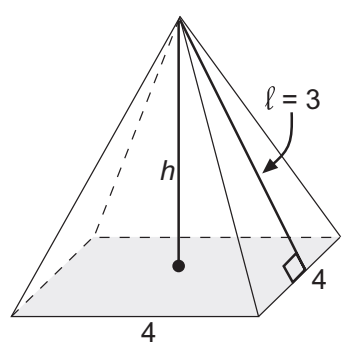
lateral area of a rectangular solid

$$L.A. = Ph$$

lateral area of a pyramid

$$L.A. = \frac{1}{2}P\ell$$

Let's go back to our example below (recopied from the previous page).



pyramid

$L.A. = \frac{1}{2}P\ell$

perimeter of the base = 4 + 4 + 4 + 4 = 16

slant height = 3

$L.A. = \frac{1}{2}$ x 16 x 3 = 24.

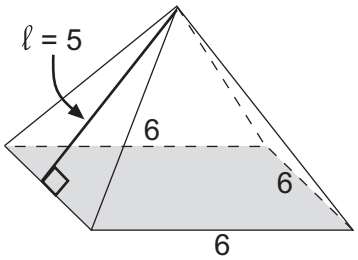
It's your turn to practice.

Practice

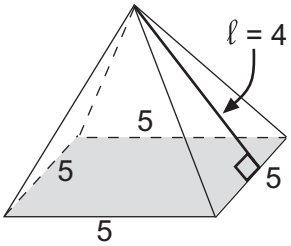
Use the **formula** *below to find the* **lateral area** *for each* **pyramid**. *Refer to pages 586 and 587 as needed.*

lateral area of a pyramid

$$L.A. = \frac{1}{2}P\ell$$

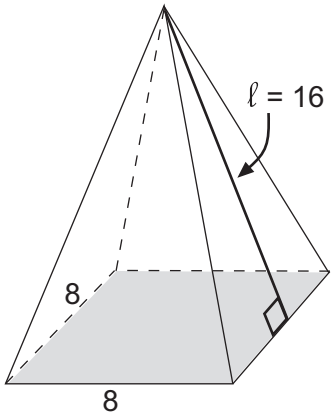
1. 

lateral area = _____

2. 

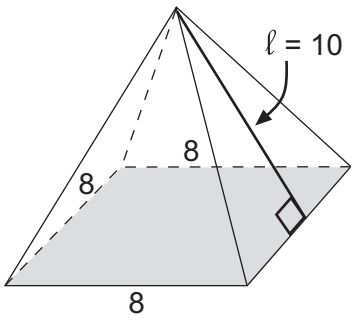
lateral area = _____

3.



lateral area = _____

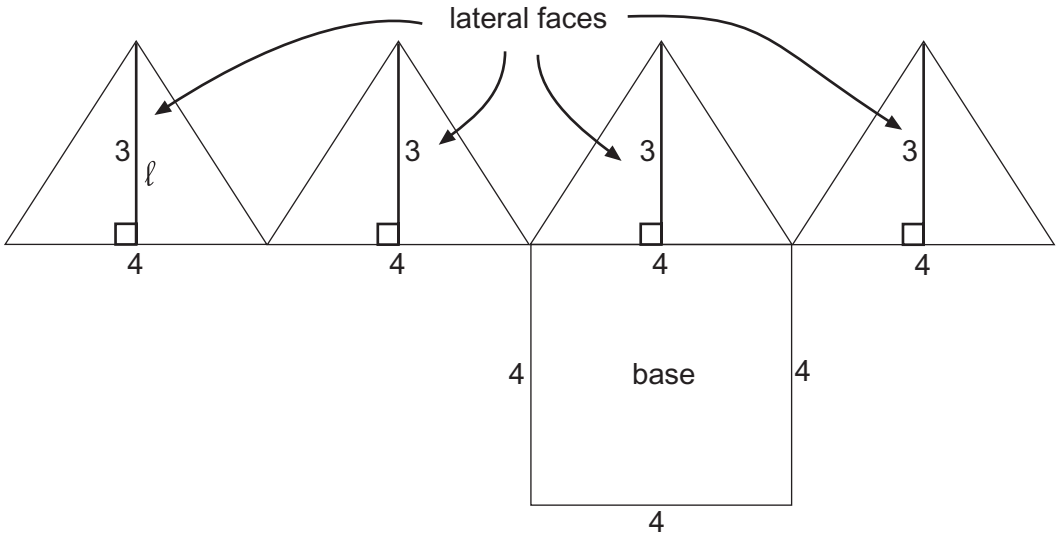
4.



lateral area = _____

Surface Area of Pyramids

Can you guess how to find the surface area of a pyramid? You guessed it—just like in a rectangular solid, you must add the base area. How many base areas? Right again: only one, because a pyramid only has one base. See the figure below.



pyramid flattened out

Therefore,

surface area of a pyramid
$S.A. = L.A. + B$

$$S.A. = L.A. + B$$
$$B = 4 \times 4$$
$$B = 16$$
$$S.A. = 24 + 16$$
$$S.A. = 40$$

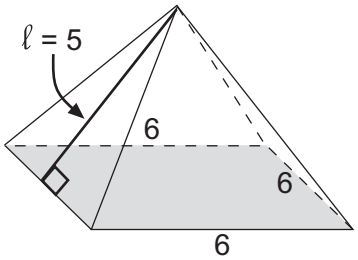
You know the drill! It's your turn to practice.

Practice

Use the **formula** *below to find the* **surface area** *for each* **pyramid** *from numbers 1-4 in the* **previous practice**. *The figures have been recopied below. Refer to page 590 as needed.*

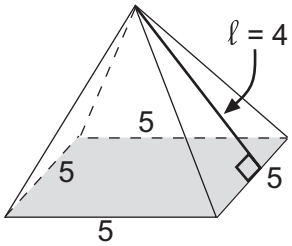
surface area of a pyramid
S.A. = L.A. + B

1.



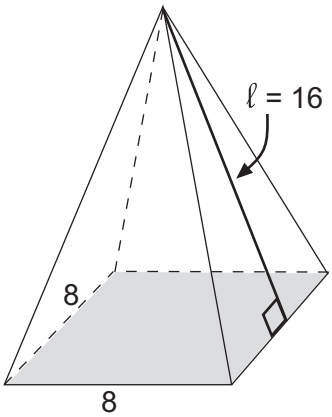
surface area = _____

2.



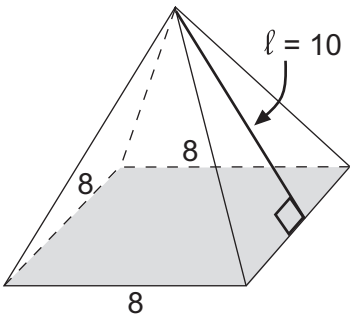
surface area = _____

3.



surface area = _____

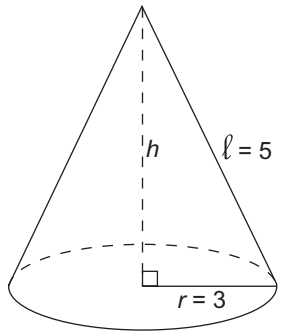
4.



surface area = _____

Areas of Cones

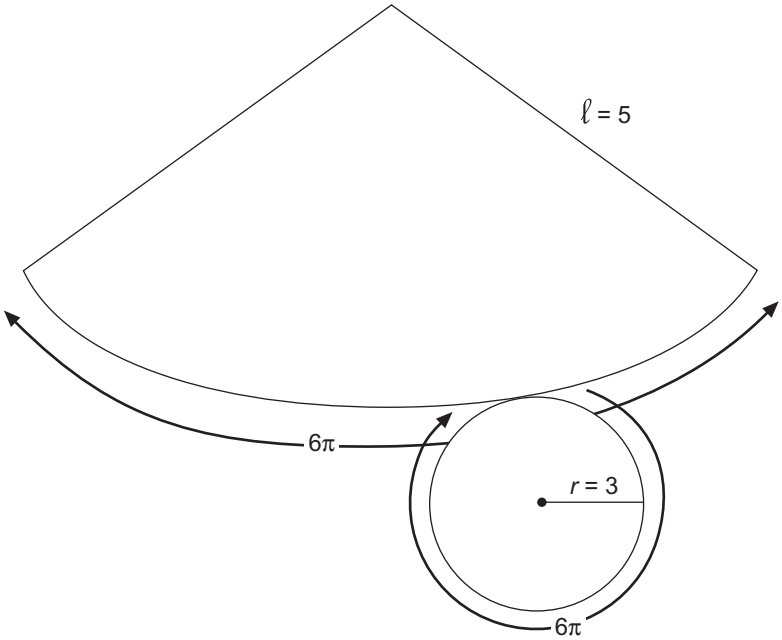
A **cone** is like a combination of a cylinder and a pyramid. Some people think of it as a pyramid with a circular base.



cone

Lateral Areas of Cones

Cut and flattened, the *cone* would look something like this.



cone flattened out

Because the cone is so much like the pyramid and basically triangular in shape, we use the same concepts for finding lateral area and surface area.

In a cone,

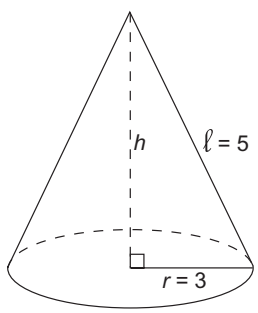
$$L.A. = \frac{1}{2}P\ell$$

We adjust the perimeter to circumference because the base is a circle, so

$$L.A. = \frac{1}{2}(2\pi r)\ell \qquad C = 2\pi r \text{ (substitute for } P\text{)}$$
$$= \pi r\ell \qquad \text{simplify } \frac{1}{2}(2)$$

lateral area of a cone
$L.A. = \frac{1}{2}(2\pi r)\ell$ **or** $L.A. = \pi r\ell$

So, for the cone illustrated below (recopied from the previous page),



cone

$$L.A. = \frac{1}{2}(2\pi \text{ x } 3) \text{ x } 5 = 15\pi$$

Surface Areas of Cones

Can you figure out the surface area?

You are correct: just add the area of the base of the cone.

surface area of a cone
$S.A. = L.A. + \pi r^2$

For a circle,

$$B = \pi r^2$$
$$= 9\pi$$
$$S.A. = L.A. + B$$
$$= 15\pi + 9\pi$$
$$= 24\pi$$

It's practice time again.

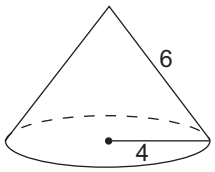
Practice

Use the **formulas** *below to find the* **lateral area** *and* **surface area** *for each* **cone** *below. Use the symbol* π *to state the* ***exact*** **areas** *of a* **cone** *in your answers. Refer to pages 593 and 594 as needed.*

lateral area of a cone
$L.A. = \pi r \ell$

surface area of a cone
$S.A. = L.A. + \pi r^2$

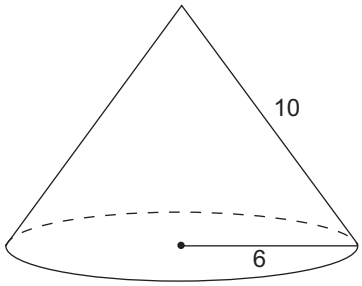
1.



lateral area = _____

surface area = _____

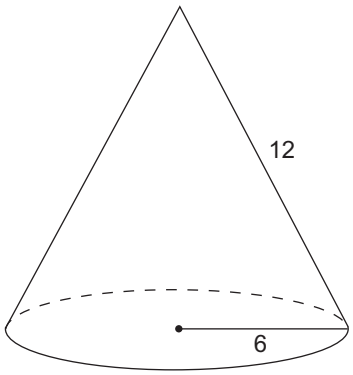
2.



lateral area = _____

surface area = _____

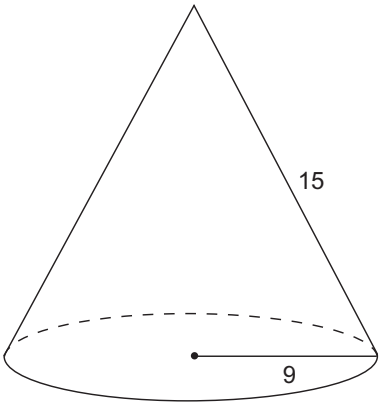
3.



lateral area = _____

surface area = _____

4.



lateral area = _____

surface area = _____

Lesson Four Purpose

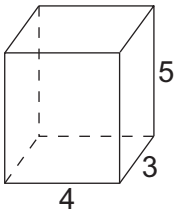
- Use concrete and graphic models to derive formulas for finding perimeter, area, surface area, circumference, and volume of two- and three-dimensional shapes, including rectangular solids, cylinders, cones, and pyramids. (MA.B.1.4.1)
- Determine the impact when changing the parameters of given functions. (MA.D.1.4.2)

Volume of Rectangular Solids and Cylinders

The **volume (*V*)** of a solid is a measure of the amount of space the solid encloses. You can think of the *volume* (V) of a rectangular solid as the area of the base or the length (l) times the width (w) times the height (h) of the solid. The same is true for a cylinder. Think of the area of the cylinder's base, which is pi (π) times the radius (r) squared times the height (h).

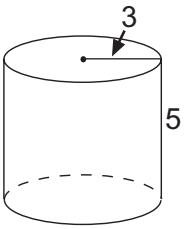
rectangular solid

Volume (V) = Bh
$V = (l \times w) \times h$
$V = (4 \times 3) \times 5$
$V = 60$



cylinder

Volume (V)= Bh
$V = (r^2\pi) \times h$
$V = (3^2\pi) \times 5$
$V = 45\pi$

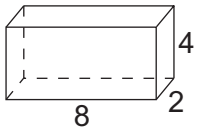


Now try the following practice.

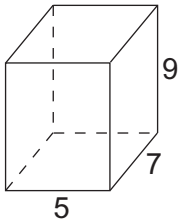
Practice

Use the **formulas** *below to find the* **volume** *of each figure. Refer to page 597 as needed.*

rectangular solid	**cylinder**
$V = Bh = (lw) \times h$	$V = Bh = (r^2\pi) \times h$

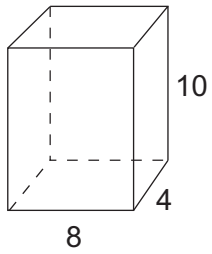
1. 

volume = _____

2. 

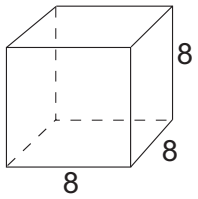
volume = _____

3.



volume = _____

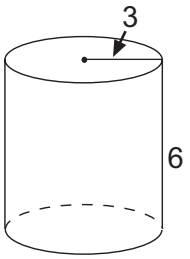
4.



volume = _____

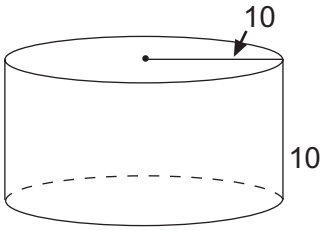
For numbers 5-10, use the symbol π *to state the* ***exact*** **volume** *of the* **cylinder** *in your answer.*

5.



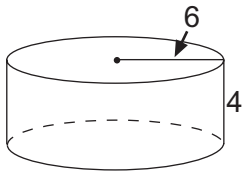
volume = _____

6.



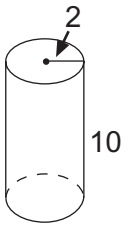
volume = _____

7.



volume = _____

8.



volume = _____

Answer the following.

9. Double each dimension for number 3 and find the volume.
   Example: 8 x 2 = 16.

   volume = _____

10. Triple each dimension for number 5 and find the volume.
    Example: 6 x 3 = 18.

    volume = _____

11. Write a rule suggesting what effect expanding the dimensions of a figure has on its volume.

    _____

    _____

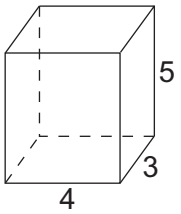
    _____

Volumes of Pyramids and Cones

When you compare a pyramid and a rectangular solid with the same base and height measurements, you can tell that the volume of the pyramid should be less than the volume of the rectangular solid. It looks smaller, but how much smaller is it? Most of us would guess $\frac{1}{2}$, but actually the pyramid is $\frac{1}{3}$ the size of the rectangular solid. With that in mind, let's compare the volume formulas for a rectangular solid and a pyramid.

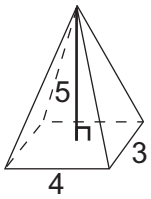
rectangular solid

Volume $(V) = Bh$
$V = (l \text{ x } w) \text{ x } h$
$V = (4 \text{ x } 3) \text{ x } 5$
$V = 60$



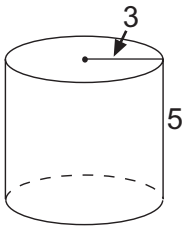
pyramid

Volume $(V) = \frac{1}{3}(Bh)$
$V = \frac{1}{3}(l \text{ x } w) \text{ x } h$
$V = \frac{1}{3}(4 \text{ x } 3) \text{ x } 5$
$V = 20$



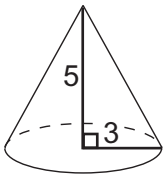
cylinder

Volume $(V) = Bh$
$V = (r^2\pi) \text{ x } h$
$V = (3^2\pi) \text{ x } 5$
$V = 45\pi$



cone

Volume $(V) = \frac{1}{3}(Bh)$
$V = \frac{1}{3}(r^2\pi) \text{ x } h$
$V = \frac{1}{3}(3^2\pi) \text{ x } 5$
$V = 15\pi$



Compare the volumes of the rectangular solid and the pyramid. Now compare the volumes of the cylinder and cone. What do you see?

Remember: The formulas for volume of cones and cylinders use the height (h) of the figure, *not* the slant height (ℓ).

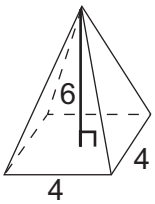
Now it's your favorite part…practice time.

Practice

Use the **formulas** *below to find the* **volume** *of each figure. Refer to page 603 as needed.*

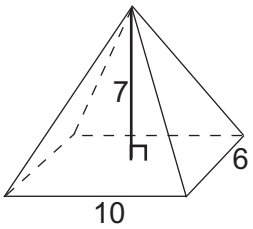
volume of a pyramid	**volume of a cone**
$V = \frac{1}{3}Bh = \frac{1}{3}(l \times w) \times h$	$V = \frac{1}{3}Bh = \frac{1}{3}(r^2\pi) \times h$

1.



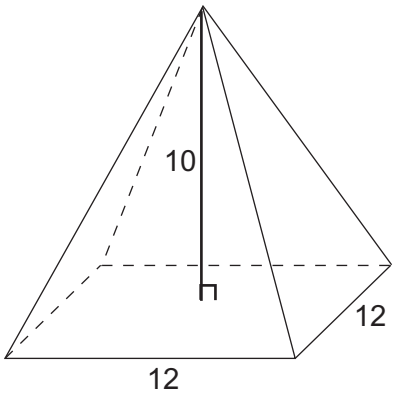
volume = _____

2.



volume = _____

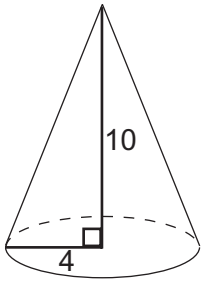
3.



volume = _____

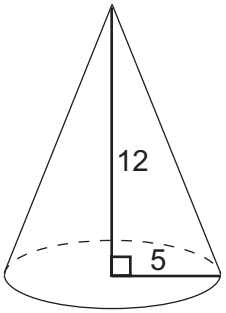
For numbers 4-6, use the symbol π *to state the* ***exact*** **volume** *of the* **cone** *in your answer.*

4.



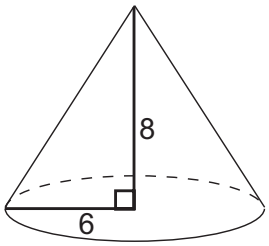
volume = _____

5.



volume = _____

6.



volume = _____

Practice

Circle the letter of the correct answer.

1. A _____ is a three-dimensional figure with one circular base in which a curved surface connects the base to the vertex.

   a. cone
   b. cylinder
   c. circle

2. A *right prism*, or a _____ , is a three-dimensional figure (polyhedron) with congruent, polygonal bases and lateral faces that are all parallelograms.

   a. pyramid
   b. cylinder
   c. rectangular solid

3. A three-dimensional figure with two parallel bases that are congruent circles is a _____ .

   a. cone
   b. cylinder
   c. rectangular solid

4. A _____ is a three-dimensional figure whose base is a polygon and whose faces are triangles with a common vertex.

   a. pyramid
   b. cone
   c. rectangular solid

5. The set of all points in a plane that are all the same distance from a given point called the center is called a _____ .

   a. cone
   b. circle
   c. pyramid

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. the sum of the areas of the faces and any curved surfaces of the figure that create the geometric solid

_____ 2. the area of the sides of a geometric solid

_____ 3. a rectangular side of a rectangular solid

_____ 4. a line segment from any point on the circle passing through the center to another point on the circle

_____ 5. the distance around a circle

_____ 6. a line segment extending from the center off a circle or sphere to a point on the circle or sphere

_____ 7. the amount of space occupied in three dimensions and expressed in cubic units

_____ 8. the area of a base (B) of a geometric solid

A. base area

B. circumference (C)

C. diameter (d)

D. lateral area ($L.A.$)

E. lateral face

F. radius (r)

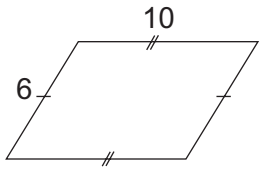
G. surface area ($S.A.$)

H. volume (V)

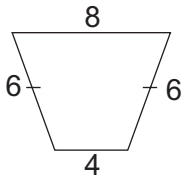
Unit Review

Find the **perimeter** *of each* **polygon**.

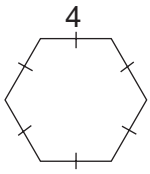
Perimeter is the distance around a polygon.

1. 

perimeter = _____

2. 

perimeter = _____

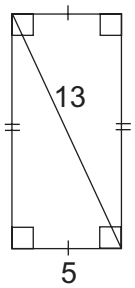
3. 

perimeter = _____

Use the **formulas** *below to find the* **area** *of each* **polygon**.

4. **rectangle**
   $A = bh$ or lw

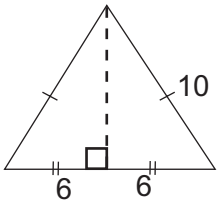
   Pythagorean theorem
   $a^2 + b^2 = c^2$



area = _____

5. **triangle**
   $A = \frac{1}{2}bh$

   Pythagorean theorem
   $a^2 + b^2 = c^2$

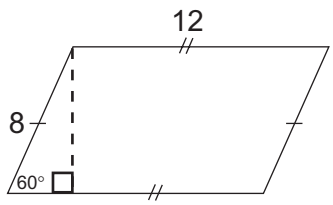


area = _____

6.

parallelogram
$A = bh$

Pythagorean theorem
$a^2 + b^2 = c^2$
or
30°-60°-90° triangle



area = _____

Remember:

30°-60°-90° triangle

- the short leg is always opposite the 30° angle and is $\frac{1}{2}$ the length of the hypotenuse
- the long leg is always opposite the 60° angle and is $\sqrt{3}$ times the length of the short leg
  or
- the length of the short leg can be found by dividing the length of the long leg by $\sqrt{3}$

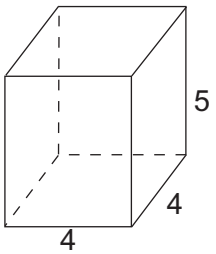
Use the **formulas** *below to find the* **lateral area**, **surface area**, *and* **volume** *of each* **solid**.

7.

lateral area of a rectangular solid
$L.A. = Ph$

surface area of a rectangular solid
$S.A. = L.A. + 2B$

volume of a rectangular solid
$V = Bh = (lw) \times h$



lateral area = _____

surface area = _____

volume = _____

8. **lateral area of a cylinder**

$L.A. = 2\pi rh$

surface area of a cylinder

$S.A. = 2\pi rh + 2\pi r^2$

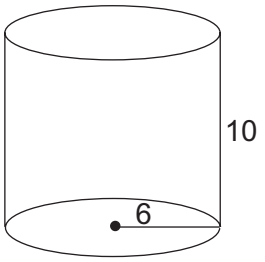
or

$S.A. = L.A. + 2B$

volume of a cylinder

$V = Bh = (r^2\pi) \times h$

Use the symbol π to state the *exact* area and volume of the cylinder in your answer.



lateral area = _____

surface area = _____

volume = _____

9. **lateral area of a pyramid**

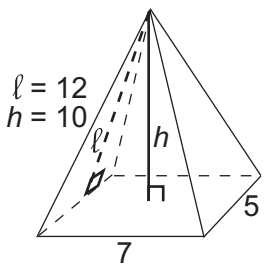
$L.A. = \frac{1}{2}P\ell$

surface area of a pyramid

$S.A. = L.A. + B$

volume of a pyramid

$V = \frac{1}{3}Bh = \frac{1}{3}(l \text{ x } w) \text{ x } h$



lateral area = _____

surface area = _____

volume = _____

10. **lateral area of a cone**

$L.A. = \pi r \ell$

surface area of a cone

$S.A. = L.A. + \pi r^2$

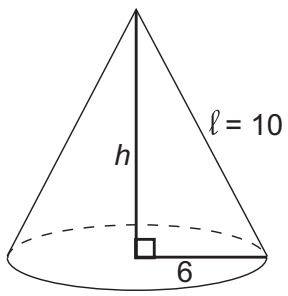
volume of a cone

$V = Bh = \frac{1}{3}(r^2\pi) \times h$

Pythagorean theorem

$a^2 + b^2 = c^2$

Use the symbol π to state the *exact* area and volume of the cone in your answer.



lateral area = _____

surface area = _____

volume = _____

Answer the following.

11. A rectangular solid has a surface area of 48. If we *double* all dimensions, what will the surface area of the new solid be?

    Answer: _____

12. A cylinder has a volume of 28π. What will be the volume of a cylinder whose dimensions are *half* as long as those of the original cylinder? Use the symbol π to state the *exact* volume of the cylinder in your answer.

    Answer: _____

Unit 10: Check Out the Statistics

This unit emphasizes the statistical concepts found in charts, experiments, central tendency, probability, permutations, and combinations.

Unit Focus

Number Sense, Concepts, and Operations

- Understand concrete and symbolic representations of real and complex numbers in real-world situations. (MA.A.1.4.3)

Data Analysis and Probability

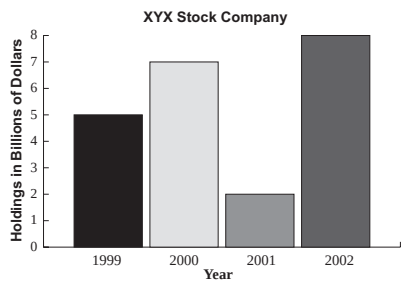
- Interpret data that has been collected, organized, and displayed in charts, tables, and plots. (MA.E.1.4.1)
- Calculate measures of central tendency (mean, median, and mode) and dispersion (range, *standard deviation, and variance*) for complex sets of data and determine the most meaningful measure to describe the data. (MA.E.1.4.2)
- Determine probabilities using counting procedures, tables, tree diagrams, and formulas for permutations and combinations. (MA.E.2.4.1)
- Design and perform real-world statistical experiments that involve more than one variable, then analyze results and report findings. (MA.E.3.4.1)

Vocabulary

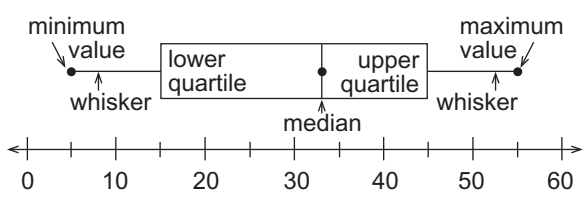
Use the vocabulary words and definitions below as a reference for this unit.

axes (of a graph) ..... the horizontal and vertical number lines used in a coordinate plane system; (singular: *axis*)

bar graph ..... a graph that uses either vertical or horizontal bars to display data



box-and-whisker plot (box plots) ..... a basic graphing tool that displays centering, spread, and distribution of a data set



box-and-whisker plot of sample set of data

canceling ..... dividing a numerator and a denominator by a common factor to write a fraction in lowest terms or before multiplying fractions

Example: $\frac{15}{24} = \frac{{}^{1}\cancel{3} \bullet 5}{2 \bullet 2 \bullet 2 \bullet \cancel{3}_{1}} = \frac{5}{8}$

chart ..... a data display that presents information in columns and rows

combination ..... an arrangement, or listing, of objects or events in which order is not accounted for

counting principle ..... if a first event has n outcomes and a second event has m outcomes, then the first event followed by the second event has $n \times m$ outcomes

data ..... information in the form of numbers gathered for statistical purposes

denominator ..... the bottom number of a fraction, indicating the number of equal parts a whole was divided into
Example: In the fraction $\frac{2}{3}$ the denominator is 3, meaning the whole was divided into 3 equal parts.

dependent events ..... two events in which the first affects the outcome of the second event

difference ..... a number that is the result of subtraction
Example: In $16 - 9 = 7$,
7 is the difference.

digit ..... any one of the 10 symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9

element or **member** ..... one of the objects in a set

equally likely ..... two or more possible outcomes of a given situation that have the same probability

even number ..... any whole number divisible by 2
Example: 2, 4, 6, 8, 10, 12 …

event ..... a possible result or outcome in probability

factorial (!) ..... a numerical operation in which a number is multiplied by all positive integers less than that number
Example: 4! = 4 x 3 x 2 x 1 = 24

formula ..... a way of expressing a relationship using variables or symbols that represent numbers

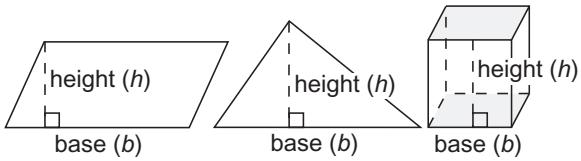
fraction ..... any part of a whole
Example: One-half written in fractional form is $\frac{1}{2}$.

frequency ..... the number of times a piece of data occurs in an interval or set of data

frequency distribution ..... a list of data organized to show how many times each item or event occurs

graph ..... a drawing used to represent data
Example: bar graphs, double bar graphs, circle graphs, and line graphs

height ..... a line segment extending from the vertex or *apex* (highest point) of a figure to its base and forming a right angle with the base or plane that contains the base



histogram ..... a bar graph that shows the frequency of data within intervals

horizontal ..... parallel to or in the same plane of the horizon

independent events ..... two events in which the outcome of the first event does *not* affect the outcome of the second event

integers ..... the numbers in the set {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

labels (for a graph) ..... the titles given to a graph, the axes of a graph, or the scales on the axes of a graph

line segment (—) ..... a portion of a line that consists of two defined endpoints and all the points in between
Example: The line segment *AB* is between point *A* and point *B* and includes point *A* and point *B*.

lower quartile ..... the median of the lower 50 percent of a set of data

mean (or average) ..... the arithmetic average of a set of numbers; a measure of central tendency

measures of central tendency ..... the mean, median, and mode of a set of data

median ..... the middle point of a set of rank-ordered numbers where half of the numbers are above the median and half are below it; a measure of central tendency

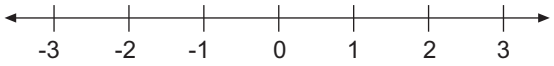
mode ..... the score or data point found most often in a set of numbers; a measure of central tendency
Example: There may be no mode, one mode, or more than one mode in a set of numbers.

multiples ..... the numbers that result from multiplying a given whole number by the set of whole numbers
Example: The multiples of 15 are 0, 15, 30, 45, 60, 75, etc.

mutually exclusive events ..... events that cannot both take place at the same time

null set (ø) or **empty set** ..... a set with no elements or members

number line ..... a line on which ordered numbers can be written or visualized



odd integers ..... any integer *not* divisible by 2
Example: {… -5, -3, -1, 1, 3, 5 …}

odd number ..... any whole number *not* divisible by 2
Example: 1, 3, 5, 7, 9, 11 …

ordered pair ..... the location of a single point on a rectangular coordinate system where the first and second values represent the position relative to the x-axis and y-axis, respectively
Example: (x, y) or (3, -4)

outcome ..... a possible result of a probability experiment

permutation ..... an arrangement, or listing, of objects or events in which order is important

positive numbers ..... numbers greater than zero

probability ..... a measure of the likelihood that a given event will occur; expressed as a ratio of one event occurring (favorable outcome) to the number of equally likely possible outcomes

range (of a set of numbers) ..... the lowest value (L) in a set of numbers through the highest value (H) in the set
Example: When the width of the range is expressed as a single number, the range is calculated as the difference between the highest and the lowest values (H – L). Other presentations show the range calculated as (H – L + 1). Depending upon the context, the result of either calculation would be considered correct.

ratio ..... the comparison of two quantities
Example: The ratio of a and b is a:b or $\frac{a}{b}$, where $b \neq 0$.

set ..... a collection of distinct objects or numbers

stem-and-leaf plot ..... a graph that organizes data by place value to compare data frequencies

Number of Goals Scored

Stem	Leaves
1	5 9
2	1 3 7 7 7
3	0 1 3 4 4 5 6 7
4	2 3 6 7 8

Key: **2** | 3 represents 23.

subset ..... a set whose elements are taken from a larger set

sum ..... the result of adding numbers together
Example: In 6 + 8 = 14,
14 is the sum.

tree diagram ..... a diagram in which all the possible outcomes of a given event are displayed

upper quartile ..... the median of the upper 50 percent of a set of data

vertical ..... at right angles to the horizon; straight up and down

whole number ..... the numbers in the set {0, 1, 2, 3, 4, …}

***x*-axis** ..... the horizontal number line on a rectangular coordinate system

***y*-axis** ..... the vertical number line on a rectangular coordinate system

Unit 10: Check Out the Statistics

Introduction

Algebra students must be able to understand the basic concepts related to statistics. Reading charts and tables, and calculating outcomes suitable to the circumstances, will be important for your future use of mathematics in real-life situations.

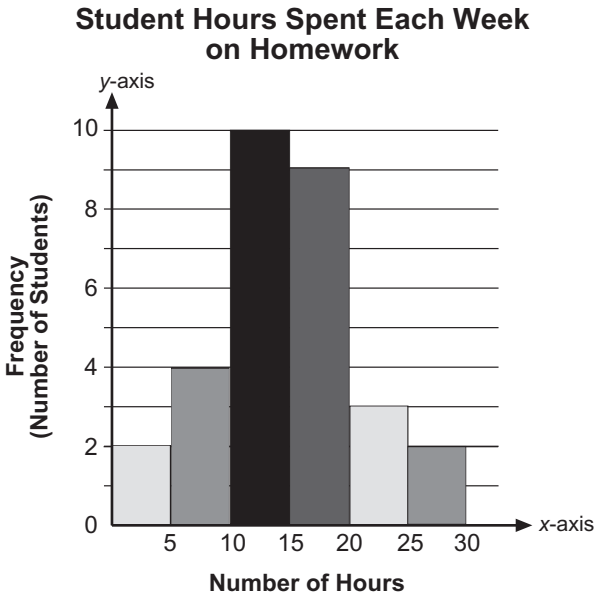
Lesson One Purpose

- Understand concrete and symbolic representations of real and complex numbers in real-world situations. (MA.A.1.4.3)
- Interpret data that has been collected, organized, and displayed in charts, tables, and plots. (MA.E.1.4.1)
- Design and perform real-world statistical experiments that involve more than one variable, then analyze results and report findings. (MA.E.3.4.1)

Histograms

Statistical **data** can be displayed in a variety of ways called **frequency distributions**. The first *frequency distribution* we will look at is a **histogram** or **bar graph**. A *histogram* involves using the *positive* sides of the ***x*-axis** and ***y*-axis** along with bars, which represent the *data*. Look at the histogram below.

Thirty students were asked to give the number of hours they spend doing homework each week. The results of the survey are shown in the histogram to the right. Notice that the **graph** is **labeled** along the *x*-axis with the *number of hours students study* and along the *y*-axis with the *number of students* (also called the **frequency**). It is also important to label each increment to clearly indicate the numbers you are recording with each bar.



histogram

- How many students studied between 5 and 10 hours per week?

  By noting the **height** of the bar between 5 and 10 on the "Number of Hours" or **horizontal (⟷)** *x*-axis, we see that 4 students studied between 5 and 10 hours per week.

- How many students studied at least 15 hours per week?

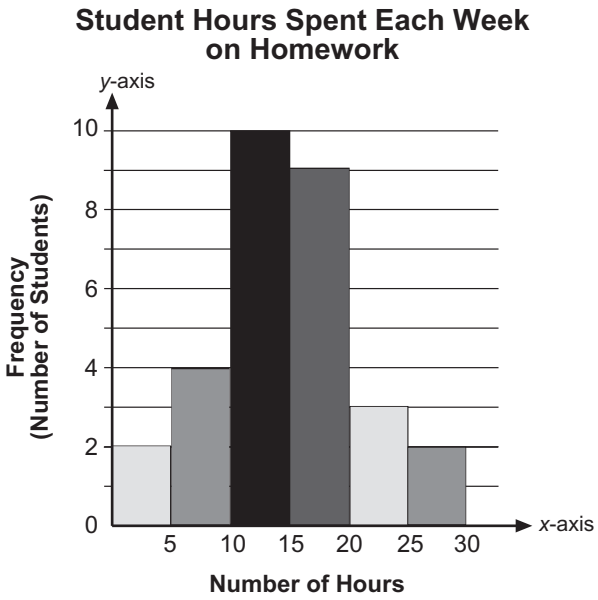
  By noting the *height* of all bars to the *right* of 15 on the *horizontal (⟷)* *x*-axis and adding those totals together, we find that the number of students who studied at least 15 hours per week is 14.

  $(9 + 3 + 2 = 14)$

Now it's your turn to practice.

Practice

Use the **histogram** *below to answer each of the following questions. Write the answer on the line provided.*



_____ 1. How many students studied *between* 20 and 25 hours per week?

_____ 2. How many students studied *less than* 20 hours per week?

_____ 3. How many students studied *more than* 10 hours per week?

_____ 4. How many students studied *between* 15 and 25 hours per week?

_____ 5. How many students studied *between* 20 and 30 hours per week?

_____ 6. How many students studied *less than* 15 hours per week?

_____ 7. How many students studied *greater than* 20 hours per week?

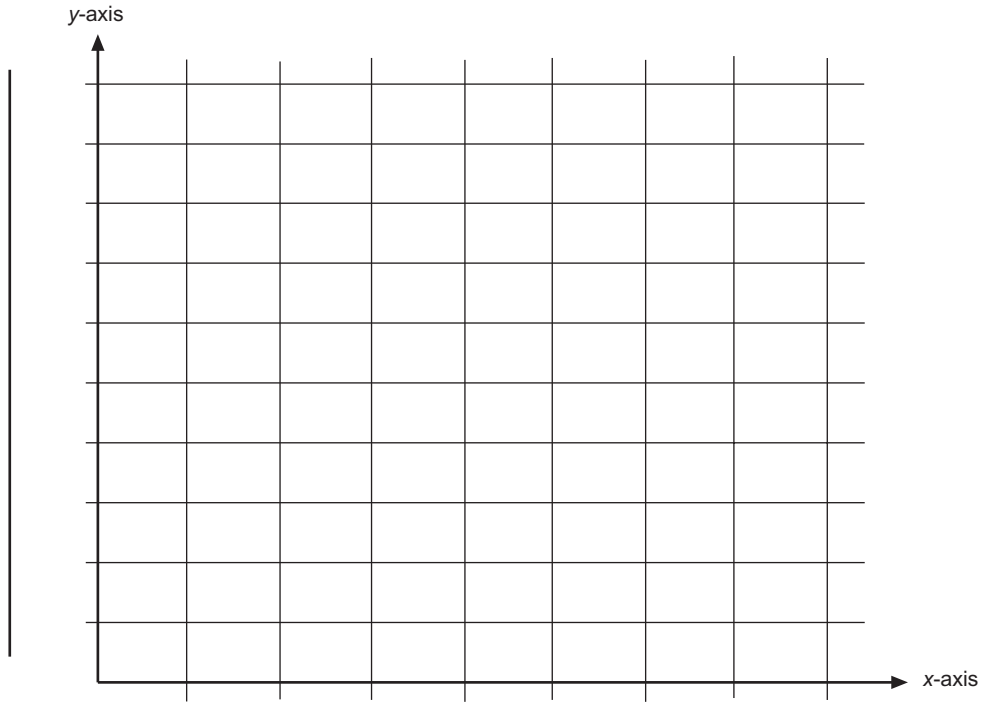
_____ 8. What was the *most popular response* among the students surveyed?

Complete the following.

9. Make a **histogram** using the information in the **chart** below, which shows the **number of hours 32 students rehearsed** to be in a play. Be sure to **label the axes** and title the histogram.

Hours Rehearsed for a Play

Number of Hours Rehearsed	Number of Students
5	4
7	6
9	9
11	7
13	4
15	2



10. Now it's your turn.

- **Compose** a **well-worded question** with **five different possible responses**.
- Then **survey** your classmates to find their **preferences** to your question, using your response choices.
- **Make** another **histogram**, this time to illustrate your findings from your survey.

Question

_____

_____

_____

Five different possible responses

_____ 1. _____

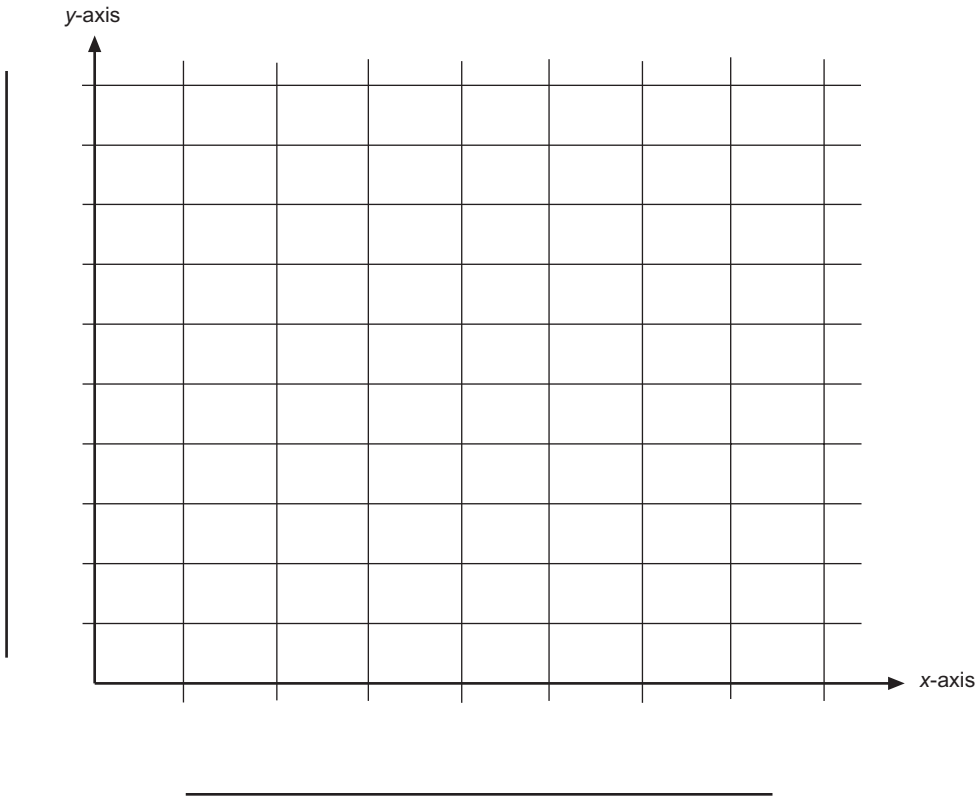
_____ 2. _____

_____ 3. _____

_____ 4. _____

_____ 5. _____

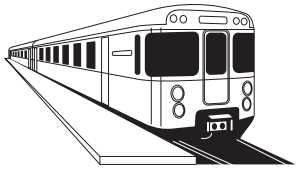
Histogram of survey responses



Stem-and-Leaf Plots

Another option for displaying statistical data is a **stem-and-leaf plot**. Let's investigate the following example.

The *stem-and-leaf plot* below represents the distribution for the following **set** of ages of people on a train.



Ages of People on a Train

26, 30, 29, 41, 35, 26, 34, 29, 35, 31, 35, 42, 46, 26, 34, 41

- The number to the *left* of the **vertical (↕)** line is the *stem*. It represents the *number of tens* from each entry above.
- Each number to the *right* of the *vertical* (↕) *line* is a *leaf*. It represents the *ones'* **digit** from each number in the list above.

Stem	Leaves
2	6, 9, 6, 9, 6
3	0, 5, 4, 5, 1, 5, 4
4	1, 2, 6, 1

Key: **3** | 4 represents 34.

Stem	Leaves
2	6, 6, 6, 9, 9
3	0, 1, 4, 4, 5, 5, 5
4	1, 1, 2, 6

Key: **3** | 4 represents 34.

Notice that the two stem-and-leaf plots above both show the same information. However, look closely. The one on the *right* lists the final results of the *leaves* in order from *least to greatest*.

Practice

Complete the **stem-and-leaf plot** *for each set of data. Arrange the* **leaves** *in order from* **least to greatest**.

1. 34, 41, 45, 43, 67, 54, 59, 62, 41

Stem	Leaves

2. 61, 64, 63, 66, 75, 54, 78, 75, 63, 75, 52

Stem	Leaves

3. 65, 76, 98, 46, 45, 99, 35, 63, 84, 67, 99, 35

Stem	Leaves

4. 76, 84, 91, 75, 79, 88, 93, 80, 93, 92, 83

Stem	Leaves

5. 68, 76, 54, 56, 65, 76, 67, 66, 76

Stem	Leaves

Lesson Two Purpose

- Understand concrete and symbolic representations of real and complex numbers in real-world situations. (MA.A.1.4.3)
- Calculate measures of central tendency (mean, median, and mode) and dispersion (range, *standard deviation, and variance*) for complex sets of data and determine the most meaningful measure to describe the data. (MA.E.1.4.2)

Central Tendency

There are several ways to interpret the **measures of central tendency** of a *set* of data. *Measures of central tendency* indicate a *center* of a set of data. **Mean**, **median**, and **mode** are all measures of central tendency.

Mean

The *mean* (or average) of a set of data is found by *adding* all the numbers in the set and then *dividing* that total by the number of **elements** or **members** in the set.

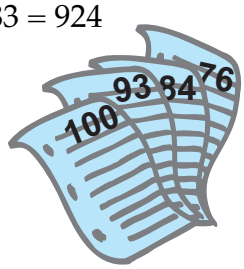
For instance, here's how to find the mean of the following set of test scores.

76, 84, 91, 75, 59, 88, 93, 100, 93, 92, 83

76 + 84 + 91 + 75 + 59 + 88 + 93 + 100 + 93 + 92 + 83 = 924

924 ÷ 11 = 84

The mean of the set of scores is 84.



Practice

Find the **mean** *of each set of numbers.*

1. 33, 54, 24, 65, 19.8, 36, 24, 65

   mean = _____

2. 12.4, 11.7, 8.5, 11, 15.3, 8.5

   mean = _____

3. 121, 435, 665, 497, 564

   mean = _____

4. 76, 54, 87.9, 112, 43.7, 25.8, 54, 76

   mean = _____

5. 98, 76, 54, 56, 65, 76

   mean = _____

6. 879, 563, 712, 904, 234

   mean = _____

7. 5430, 4130, 2310, 4378

   mean = _____

8. 10, 6, 4, 6.8, 9, 4.5, 10

   mean = _____

Median

To find the *median* of the same set of scores below from page 636, we put them in order from *least to greatest* and find the one in the *middle*.

59, 75, 76, 83, 84, 88, 91, 92, 93, 93, 100

There are 11 scores, and the middle one is 88. Therefore, the median of these scores is 88.

Note: If the number of elements in your set of data is **even**, then you must find the *mean of the two middle numbers*.

Practice

Find the **median** *of each set of numbers.*

1. 33, 54, 24, 65, 19.8, 36, 24, 65

   median = _____

2. 12.4, 11.7, 8.5, 11, 15.3, 8.5

   median = _____

3. 121, 435, 665, 497, 564

   median = _____

4. 76, 54, 87.9, 112, 43.7, 25.8, 54, 76

   median = _____

5. 98, 76, 54, 56, 65, 76

   median = _____

6. 879, 563, 712, 904, 234

   median = _____

7. 5430, 4130, 2310, 4378

   median = _____

8. 10, 6, 4, 6.8, 9, 4.5, 10

   median = _____

Mode

We can also find the *mode* of the set of scores. Use the same set of scores below from page 639. The mode is the *number that appears most often*.

59, 75, 76, 83, 84, 88, 91, 92, 93, 93, 100

The mode of the scores above is 93.

Note: If *no* number appears more often than the others, then we say that the set has *no* mode.

Practice

Find the **mode** *of each set of numbers. Write* **none** *if the set has* **no** *mode.*

1. 33, 54, 24, 65, 19.8, 36, 24, 65

   mode = _____

2. 12.4, 11.7, 8.5, 11, 15.3, 8.5

   mode = _____

3. 121, 435, 665, 497, 564

   mode = _____

4. 76, 54, 87.9, 112, 43.7, 25.8, 54, 76

   mode = _____

5. 98, 76, 54, 56, 65, 76

   mode = _____

6. 879, 563, 712, 904, 234

   mode = _____

7. 5430, 4130, 2310, 4378

   mode = _____

8. 10, 6, 4, 6.8, 9, 4.5, 10

   mode = _____

Range

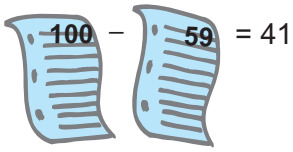
Another fact we could find about the set of scores is the **range**. Use the same set of scores below from page 639. The *range* is the **difference** *between* the largest and smallest numbers in the list.

59, 75, 76, 83, 84, 88, 91, 92, 93, 93, 100

In this case, the largest score is 100 and the smallest is 59.

100 – 59 = 41.

So, the range is 41.



Practice

Find the **range** *of each set of numbers.*

1. 33, 54, 24, 65, 19.8, 36, 24, 65

   range = _____

2. 12.4, 11.7, 8.5, 11, 15.3, 8.5

   range = _____

3. 121, 435, 665, 497, 564

   range = _____

4. 76, 54, 87.9, 112, 43.7, 25.8, 54, 76

   range = _____

5. 98, 76, 54, 56, 65, 76

   range = _____

6. 879, 563, 712, 904, 234

   range = _____

7. 5430, 4130, 2310, 4378

   range = _____

8. 10, 6, 4, 6.8, 9, 4.5, 10

   range = _____

Choosing a Measure of Central Tendency

You can use a stem-and-leaf plot to help find mean, median, mode, and range. See the example below.

Stem	Leaves
2	6, 6, 9, 9
3	0, 1, 4, 4, 5, 5, 5
4	1, 1, 2, 6

Key: **3** | 4 represents 34.

mean = 34.2667	Add the data (514) and divide by 15.
median = 34	Find the middle value of the data.
mode = 35	Find the most frequent value of the data.
range = 20	Find the difference between the highest and the lowest value of the data—46 – 26.

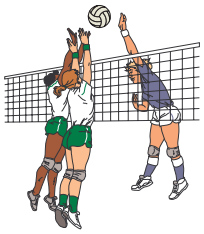
We use the different type averages discussed above in a variety of ways, depending upon what we are trying to do with the statistical information.

Think about the situations in the following practice.

Practice

Answer the following.

1. The five girls on a volleyball team measured their heights. Their results in inches were 68″, 66″, 68″, 72″, and 70″. If their coach wants to intimidate other teams by printing their average height, which type of average is she most likely to use—mean, median, or mode? Explain your answer.



_____

_____

_____

2. What if the heights on the team above were 59″, 63″, 68″, 70″, and 78″? Would your answer change? Why?

_____

_____

_____

3. If you own a shoe store and need to order new shoes to stock up for the season, would you be more likely to use the mean, median, or mode concept to determine how many shoes of certain sizes to order? Why?

_____

_____

_____

Box-and-Whisker Plots

Now that we know how to determine the median of a set of data, we can learn a third way to display data. We can use a **box-and-whisker plot**. It is called *box-and-whisker plot* because it looks like a box with a whisker at each end.

This type of diagram clearly shows the median and range of the distribution, as well as, the first and third quartile. The *first quartile*, or **lower quartile**, is the median of the *lower* half of the data. The *third quartile*, or **upper quartile**, is the median of the *upper* half of the data.

Finding the Median of the Set of Data

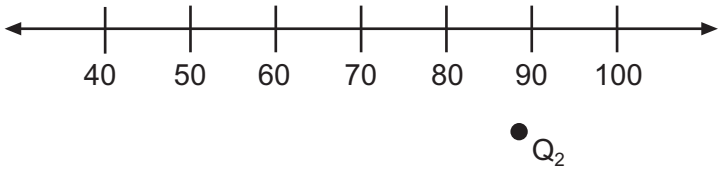
To draw a box-and-whisker plot, we must first find the median of the entire set of data. Let's use the following test scores to illustrate.

59, 75, 76, 83, 84, 88, 91, 92, 93, 93, 100

median of set of data

The median of this set of test scores is 88. We use a dot and Q_2 to show this number representing the median. See the **number line** below.

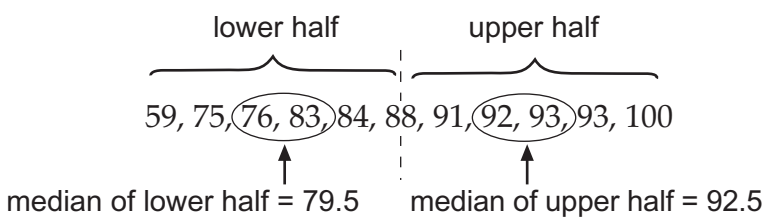
Test Scores

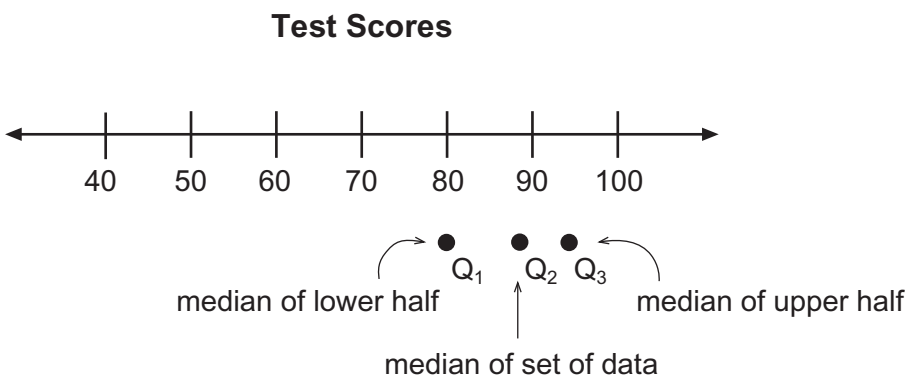


Finding the Median of the Lower and Upper Half of the Data

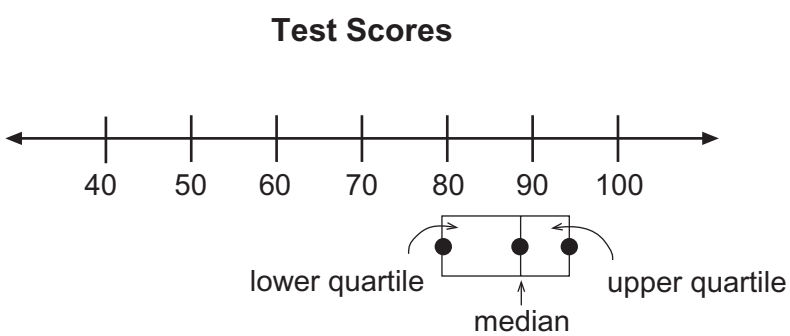
Next, we find the median of the lower half of the data, including the median. This is the first quartile, shown by Q_1 on the number line below at 79.5.

Then find the median of the upper half of the data, including the median. This is the third quartile, shown by Q_3 (below) at 92.5.

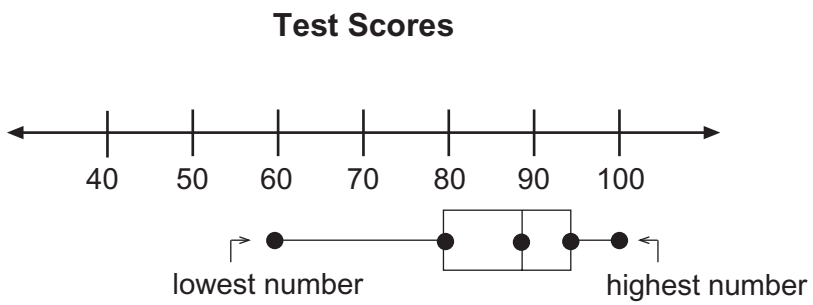




Now draw a narrow rectangular box with ends on the two quartile dots Q_1 and Q_3, with a vertical **line segment (—)** through the median Q_2, as shown below.



Last, we draw the whiskers from Q_1 and Q_3 to the points 59 and 100, which represent the lowest and highest numbers in the range of data. This is a box-and-whisker plot.



Now, try a few in the following practice.

Practice

Draw a **box-and-whisker plot** *for each set of data.*

1. 23, 32, 41, 44, 56, 59, 62

Q_1 = _____

Q_2 (median) = _____

Q_3 = _____

2. 56, 67, 68, 72, 75, 90, 93

Q_1 = _____

Q_2 (median) = _____

Q_3 = _____

3. 51, 58, 67, 73, 76, 82

Q_1 = _____

Q_2 (median) = _____

Q_3 = _____

Practice

Use the list below to write the correct term for each definition on the line provided.

box-and-whisker plot (box plots)	**median**
lower quartile	**mode**
mean (or average)	**range (of a set of numbers)**
measures of central tendency	**upper quartile**

_____ 1. the mean, median, and mode of a set of data

_____ 2. the score or data point found most often in a set of numbers

_____ 3. the median of the *lower* 50 percent of a set of data

_____ 4. the arithmetic average of a set of numbers

_____ 5. the middle point of a set of rank-ordered numbers where half of the numbers are above the median and half are below it

_____ 6. the median of the *upper* 50 percent of a set of data

_____ 7. a basic graphing tool that displays centering, spread, and distribution of a data set

_____ 8. the lowest value (L) in a set of numbers through the highest value (H) in the set

Lesson Three Purpose

- Determine probabilities using counting procedures, tables, tree diagrams, and formulas for permutations and combinations. (MA.E.2.4.1)

Counting Principles

There are processes that help us determine the number of choices we have in certain situations. These processes are known as **counting principles**. One of the *counting principles* involves multiplication and another involves addition. We'll look at each one.

The first principle says the following:

> If one selection can be made in x ways and, for each of these selections, another selection can be made in y ways, then the total number of possibilities is found by multiplying x and y together.

Let's look at an example.

Suppose we needed to answer the following question:

> How many **odd** 2-digit **positive integers** *less than* 70 are there?

To answer this question, it is helpful to draw 2 boxes, like this:

The *first box* below represents the *number of choices* for the *tens digit*.

- Because we are looking for 2-digit numbers, our choices for the *first box* are 1, 2, 3, 4, 5 and 6.
  - 7 is *not* a choice because we are only looking for numbers *less than* 70.
  - Also, 0 is *not* a choice because that would not help us to have a 2-digit number.
- So, from the above list, we see that there are 6 choices for the *first box*.

$\boxed{6}\ \boxed{}$

Now, we must determine the *number of choices* we have for the *second box*.

- Remember, we are looking for *odd integers*, so the choices for the *ones digit* are 1, 3, 5, 7, and 9.
- Therefore, there are 5 choices for the *second box*. See below.

$\boxed{6} \times \boxed{5}$

To find the number of possibilities, we multiply 6 x 5. There are 30 *odd* 2-digit *positive* integers *less than* 70.

Maybe we should look at another example.

How many 3-digit positive *even* integers *less than* 500 are there?

For this problem, we'll need 3 boxes because we are looking for 3-digit integers.

$\square \quad \square \quad \square$

To find the answer, we need to look for the number of *even* integers from 100 through 499.

- So, for the *first box*, we need to put the number of choices we have for the *hundreds digit*.
- Since the choices are 1, 2, 3, and 4, we have 4 choices for the *first box*.

  $\boxed{4} \quad \square \quad \square$

Now we must select the number of choices for the *tens box*.

- We could have 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
- So, we have 10 choices for the *second box*.

  $\boxed{4} \quad \boxed{10} \quad \square$

Finally, we select the number of choices for the *ones box*. Because we are only looking for *even* integers, we can only use *even* digits in the *ones box*.

- So, our choices here are 0, 2, 4, 6, and 8.
- Therefore, there are 5 choices for the *final box*.

  $\boxed{4} \times \boxed{10} \times \boxed{5}$

When we multiply 4 x 10 x 5, we see that there are 200 positive *even* integers *less than* 500.

$\boxed{4} \times \boxed{10} \times \boxed{5} = 200$

Now, you can try this on the following practice.

Practice

Answer the following.

1. How many *odd* 2-digit positive odd integers *less than* 50 are there?

   Answer: _____

   ☐ x ☐

2. How many *odd* 2-digit positive odd integers *greater than* 50 are there?

   Answer: _____

   ☐ x ☐

3. How many *even* 3-digit positive integers can be written using the numbers 3, 4, 5, 6, and 7?

   Answer: _____

   ☐ x ☐ x ☐

4. How many *odd* 2-digit positive integers can be written using the numbers 3, 4, 5, 6, and 7?

   Answer: _____

   ☐ x ☐

5. How many *even* 2-digit positive integers can be written using the numbers 2, 4, 5, and 8?

   Answer:_____

   $\square \times \square$

6. How many *odd* 2-digit positive integers can be written using the numbers 2, 4, 5, and 8?

   Answer: _____

   $\square \times \square$

Study the following **example**, *then answer the rest.*

Example:

In how many ways can a 5-question True/False quiz be answered if every question *must be answered*?

We will use 5 boxes, one for each question. Each question has 2 choices.

$\boxed{2} \times \boxed{2} \times \boxed{2} \times \boxed{2} \times \boxed{2}$

The total number of ways this quiz can be answered is 32. Are you surprised?

$\boxed{2} \times \boxed{2} \times \boxed{2} \times \boxed{2} \times \boxed{2} = 32$

7. In how many ways can the same 5-question True/False quiz be answered if it is okay to *leave questions unanswered*? (**Hint:** You now have 3 choices for each question.)

   Answer: _____

8. In how many ways can an 8-question True/False quiz be answered if every question *must be answered*?

   Answer: _____

9. In how many ways can a 5-question True/False quiz be answered if it is okay to *leave questions unanswered*?

   Answer: _____

10. In how many ways can a 5-question multiple choice quiz be answered if there are 4 choices for each question, and every question *must be answered*?

    Answer: _____

11. In how many ways can a 5-question multiple choice quiz be answered if there are 4 choices for each question, and is it okay to *leave questions unanswered*?

    Answer: _____

Another Counting Principle

Now let's look at another counting principle. This concept deals with **mutually exclusive events**. Two **events** are considered to be *mutually exclusive* if the occurrence of one *prevents* the other from happening. For example, a number can be even *or* odd, but *not* both.

This counting principle states that, if the possibilities that we are considering can be grouped into mutually exclusive cases, then to find the total numbers of possibilities, we must *add* the number of possibilities for each case.

Let's look at a problem to help us interpret this rule.

> How many *even* positive integers *less than* 200 are there?

Notice that this problem did not specify how many digits the numbers could be, so we have to find the possibilities for all three cases,

- the 1-digit numbers
- the 2-digit numbers
- the 3-digit numbers

and then add them together.

First, let's look at the possibilities for 1-digit numbers. There are no choices for the tens or hundreds digits, and the choices for the *ones digit* are 0, 2, 4, 6, and 8. So there are *5 choices*.

$\boxed{-} \times \boxed{-} \times \boxed{5}$

Now look at the choices for 2-digit numbers. Again, there are no choices for the hundreds box. The tens box could contain 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9, so there are *10 choices*. And, because we specified an *even* number, the *only* choices for the *ones box* remain 0, 2, 4, 6, and 8—or *5 choices*.

$\boxed{-} \times \boxed{10} \times \boxed{5}$

Finally, we decide on the number of possibilities for 3-digit numbers. We *cannot* use 0 in the hundreds place, so there is *only 1 choice* for the *hundreds box*.

$$\boxed{1} \times \boxed{10} \times \boxed{5}$$

Now, we go back and use the *first* fundamental counting principle to find the possibilities for each case. Multiplying across each row of boxes, we see that there are 5 one-digit numbers, 50 two-digit numbers, and 50 three-digit numbers.

$$\begin{array}{ccccc} \boxed{1} & \times & \boxed{10} & \times & \boxed{5} \\ 50 & & 50 & & 5 \end{array}$$

To find the total number of even positive integers *less than* 200, we add the three numbers together.

$$\begin{array}{ccccc} \boxed{1} & \times & \boxed{10} & \times & \boxed{5} \\ \multicolumn{5}{c}{50 + 50 + 5 = 105} \end{array}$$

$$5 + 50 + 50 = 105$$

Now try the following practice.

Practice

Answer the following.

1. How many *odd* positive integers *less than* 50 are there?

   Answer: _____

2. How many *odd* positive integers *greater than* 50 and less than 140 are there?

   Answer: _____

3. How many *even* positive integers *less than* 100 can be written using the numbers 3, 4, 5, 6, and 7?

   Answer: _____

4. How many *odd* positive integers *less than* 500 can be written using the numbers 3, 4, 5, 6, and 7?

   Answer: _____

5. How many *even* positive integers *less than* 600 can be written using the numbers 2, 4, 5, and 8?

   Answer: _____

6. How many *odd* positive integers *less than* 300 can be written using the numbers 2, 4, 5, and 8?

   Answer: _____

Tree Diagrams

We can also use a **tree diagram** with the counting principle. This is helpful when there are multiple arrangements to consider. For example:

> Karen has three skirts: a blue one, a black one, and a red one. She also has four blouses: white, green, yellow, and pink. She has a pair of sandals and a pair of boots. How many different outfits can Karen make with these items? We'll use a *tree diagram* to find out.

Skirts	Blouses	Shoes	Outcomes
blue	white	sandals	1
		boots	2
	green	sandals	3
		boots	4
	yellow	sandals	5
		boots	6
	pink	sandals	7
		boots	8
black	white	sandals	9
		boots	10
	green	sandals	11
		boots	12
	yellow	sandals	13
		boots	14
	pink	sandals	15
		boots	16
red	white	sandals	17
		boots	18
	green	sandals	19
		boots	20
	yellow	sandals	21
		boots	22
	pink	sandals	23
		boots	24

3	x	4	x	2	=	24

By counting the far *right* column, we see that Karen can make 24 different outfits. We could also have determined this answer by multiplying the number of skirts by the number of blouses and then multiplying by the number of pairs of shoes.

$3 \times 4 \times 2 = 24.$

Okay, now you try the following in the next practice.

Practice

Answer the following.

1. A restaurant serves 4 main dishes, 2 types of salad, and 3 different desserts. How many different meal choices could you make by choosing one main dish, one salad, and one dessert?

   Answer: _____

2. How many ways can you arrange 6 different letters, if you use each one only once?

   Answer: _____

3. How many different computer combinations can be packaged if there is a choice of 3 hard drives, 4 monitors, and 3 printers?

   Answer: _____

4. How many four-letter passwords can be made if the first letter must be A *or* B, the second letter must be X, Y, *or* Z , the third letter must be a Q, and the final letter must be a vowel (A, E, I, O, *or* U)?

   Answer: _____

Lesson Four Purpose

- Determine probabilities using counting procedures, tables, tree diagrams, and formulas for permutations and combinations. (MA.E.2.4.1)

Probability

Probability is the likelihood that an event will occur. It is expressed as a **ratio** or **fraction**. To find *probability*, we use the following **formula**.

$$\text{Probability} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

For example, if we wanted to know the probability of rolling a 3 on a regular 6-sided *die* (which is singular for *dice*), we would know that, since there is *only* one 3 on the die, the number of favorable **outcomes** is 1. Also, because there are 6 different **equally likely** possible *outcomes*, the probability of rolling a 3 is $\frac{1}{6}$.

What would be the probability of rolling an **odd number** on a regular six-sided die? Since there are 3 *odd numbers* on a die with 6 sides, the *ratio* would be as follows:

$$\text{Probability} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

What would be the probability of rolling a 2 *or* a 3? Because there are now 2 outcomes considered favorable and still 6 total outcomes, the probability is $\frac{2}{6} = \frac{1}{3}$.

Try the following before we continue!

Practice

Answer the following. Refer to page 669 as needed.

When flipping **1 regular coin**, what is the **probability of the coin** landing

1. heads up? _____
2. tails up? _____
3. tails down? _____
4. heads *or* tails up? _____
5. heads *and* tails up? _____

When rolling **1 regular die**, what is the **probability of rolling**

6. a 5? _____
7. an even number? _____
8. a 4 *or* a 5? _____
9. a 4 *and* a 5? _____
10. a **multiple** of 3? _____

Remember: *Multiples* are the numbers that result from multiplying a given **whole number** {0, 1, 2, 3, 4, …} by the set of *whole numbers*.
Example: The multiples of 15 are 0, 15, 30, 45, 60, 75, etc.

11. a 0? _____
12. an odd number *or* a 2? _____
13. a 2, 3, 4 *or* 6? _____
14. an even number *or* an odd number? _____
15. a 7? _____

Possible Outcomes Using Two Coins

Now let's consider what happens when we toss 2 coins. Because each coin has 2 faces, there are more possible outcomes to consider.

Look at the chart below, which shows all the possible outcomes when we toss the 2 coins.

Possible Outcomes of Tossing 2 Coins

Coin 1	Coin 2
H	H
H	T
T	H
T	T

You'll notice there are now 4 possible outcomes. So, the outcome of tossing exactly one coin heads up occurs 2 times in the chart, so the probability of tossing exactly one head is $\frac{2}{4} = \frac{1}{2}$.

What is the probability of tossing *at least* one head? Is that different? Yes, because *at least* one means *one* **or** *more than one*. So the probability of tossing at least one head is $\frac{3}{4}$.

Practice

Answer the following. Refer to page 671 as needed.

Look at the **chart** below to see the **possible outcomes** when tossing **2 coins**.

Possible Outcomes of Tossing 2 Coins

Coin 1	Coin 2
H	H
H	T
T	H
T	T

When tossing **2 coins**, what is the **probability that you will toss**

1. 2 heads? _____
2. *at least* 1 head? _____
3. 1 head *and* 1 tail? _____
4. 2 tails? _____
5. *at least* three heads? _____
6. no tails? _____

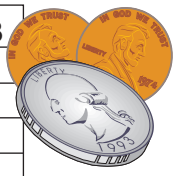
Practice

Answer the following.

Now, look at the chart below to see the **possible outcomes** when tossing **3 coins**.

Possible Outcomes of Tossing 3 Coins

Coin 1	Coin 2	Coin 3
H	H	H
H	H	T
H	T	H
H	T	T
T	H	H
T	H	T
T	T	H
T	T	T



When tossing **3 coins**, find the **probability that you will toss**

1. 3 heads. _____
2. 2 heads. _____
3. *at least* 2 heads. _____
4. 1 tail. _____
5. *at least* 1 tail. _____
6. 4 tails. _____
7. *all* heads **or** *all* tails. _____
8. *at least* 2 heads **or** *at least* 2 tails. _____

Practice

Answer the following.

When rolling 2 dice, it helps to think of the outcomes as **ordered pairs**. For instance, if you roll a 3 on the first die and a 5 on the second die, you might record it as (3, 5).

1. Fill in the rest of the **chart** below to see **all the possibilities** for rolling 2 dice.

Possible Outcomes of Rolling 2 Dice

(1, 1)	(2, 1)	(3, 1)			(6, 1)
		(3, 2)		(5, 2)	
	(2, 3)			(5, 3)	
			(4, 4)		(6, 4)
(1, 5)					
(1, 6)			(4, 6)		(6, 6)

Do you see that there are now **36 possible outcomes**?

When tossing **2 dice**, find the **probability that**

2. *exactly* one die shows a 5? _____
3. *at least* one die shows a 2? _____
4. the **sum** of the numbers showing on the dice is 7? _____
5. the *sum* of the numbers showing on the dice is *less than* 5? _____
6. the *sum* of the numbers showing on the dice is *greater than* 8? _____
7. the *sum* of the numbers showing on the dice is *greater than* 12? _____

Independent and Dependent Events

Independent Events

When the probability of one event does *not* affect the probability of the next event, we say that the two events are **independent events**. When the events are *independent*, we *multiply* the *individual probabilities* to find the probability that *both* events will occur.

For example:

A bag contains

- 2 white marbles
- 2 red marbles
- 1 blue marble.

We plan to randomly draw one marble, then replace it, and then randomly draw again. What is the probability that we will draw a *red* marble *both times*?

These two drawings are independent because the outcome of the first selection does *not* have any effect on the outcome of the second selection.

1. The probability of drawing a *red* marble on the *first draw* is $\frac{2}{5}$.
2. The probability of drawing a red marble the *second time* is also $\frac{2}{5}$.

   Remember: We replaced the first marble we drew.

3. The probability of drawing a red marble *both times* is $\frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$.

Dependent Events

Look at the same example, now with a slight twist.

The bag still contains

- 2 white marbles
- 2 red marbles
- 1 blue marble.

However, this time we plan to randomly draw one marble, and *not* replace it, and then randomly draw again. What is the probability that we will draw a *red* marble both times?

These two drawings are **dependent events** because the outcome of the first selection *does* have an effect on the outcome of the second selection.

1. The probability of drawing a *red* marble on the *first draw* is still $\frac{2}{5}$.

2. The probability of drawing a *red* marble the *second time* is now $\frac{1}{4}$.

   Note: This is because, if we drew a red marble the *first time*, there are only 4 marbles left and *only* 1 of those is red.

3. Therefore, the probability of drawing a *red* marble *both times* is $\frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10}$.

Let's look at one more twist on that bag of marbles.

The bag still contains

- 2 white marbles
- 2 red marbles
- 1 blue marble.

We plan to randomly draw one marble. What is the probability that we draw a *red* **or** a *blue* marble?

1. The probability of drawing a *red* marble is still $\frac{2}{5}$.

2. The probability of drawing a *blue* marble is $\frac{1}{5}$, because there is *only* 1 blue marble in the bag.

3. To find the probability of drawing a *red* **or** a *blue* marble is $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$.

Remember:

- The word **or** implies more choices, so you **add** the probabilities together.
- The word **and** implies fewer choices, so you **multiply** because this tends to make the **denominator** (possible outcomes) *greater*, thus *reducing* the probability.

Practice

Answer the following.

A new bag of marbles contains

- 3 yellow
- 5 green
- 4 orange
- 8 blue marbles.

Remember:

- The word **or** implies more choices, so you **add** the probabilities together.
- The word **and** implies fewer choices, so you **multiply** because this tends to make the **denominator** (possible outcomes) *greater*, thus *reducing* the probability.

1. What is the probability of drawing 2 *green* marbles if you draw once and replace the marble, and then draw again?

   Answer: _____

2. What is the probability of drawing 2 *blue* marbles if you draw once and do *not* replace the marble, and then draw again?

   Answer: _____

3. What is the probability of drawing 3 *yellow* marbles if you draw three times and replace the marble each time?

   Answer: _____

4. What is the probability of drawing 3 *yellow* marbles if you draw three times but do *not* replace the marble each time you draw?

   Answer: _____

5. What is the probability of drawing a *green* **or** an *orange* marble on the first draw?

   Answer: _____

6. What is the probability of drawing a *blue* marble on the first draw, replacing it, and then drawing an *orange* marble?

   Answer: _____

7. What is the probability of drawing a *blue* marble on the first draw, keeping it, and then drawing an *orange* marble?

   Answer: _____

8. What is the probability of drawing a *different color* marble each of four draws if you *keep* the marble each time?

   Answer: _____

9. What is the probability of drawing a *yellow*, *orange*, **or** *blue* marble on one draw?

   Answer: _____

10. What is the probability of drawing a *green* **or** *blue* marble on the first draw, replacing it, and then drawing a *yellow* marble on the second draw?

    Answer: _____

Lesson Five Purpose

- Determine probabilities using counting procedures, tables, tree diagrams, and formulas for permutations and combinations. (MA.E.2.4.1)

Permutations

Sometimes it is helpful to be able to arrange some of the elements in a set. If we want to arrange those elements in a definite order we use a **permutation**.

Before we can use the rules for *permutations*, we must learn a new mathematical symbol. That symbol looks like an exclamation mark (!) and is read as **factorial**. The *factorial* of 6 is written as 6! (which reads as 6 factorial). That doesn't mean we say 6 with extra emphasis; it means we multiply 6 by *all* the *positive integers less than* 6.

For example:

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1, \text{ or } 720.$$

We can use other mathematical operations along with the factorial symbol.

$$4!5! = 4 \times 3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1 = 2{,}880$$

$$\frac{6!}{4!} = \frac{6 \bullet 5 \bullet \cancel{4} \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}}{\cancel{4} \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}} = 30$$ Notice how several of the factors **cancel**.

This can save you lots of time. Your calculator may have a "!" key. Explore and figure out how to use it.

Important: Do *not* reduce the *fraction* before you do the factorial part! You may get the wrong answer.

Now, try the practice on the following page.

Practice

Answer the following. Refer to page 681 as needed.

Remember: Do *not* reduce the fraction before you do the factorial part.

1. 5! =

2. 4! =

3. 7! =

4. 3!4! =

5. 6!3! =

6. $5!5! =$

7. $\frac{5!}{6!} =$

8. $\frac{5!}{3!} =$

9. $\frac{3!4!}{5!} =$

10. $\frac{7!5!}{6!} =$

Possible Permutations

Now let's go back to that idea of permutations. Take the set {c, a, t}. The possible permutations of the set using *all* three letters are as follows.

cat
cta
atc
act
tac
tca

Notice that there are 6 different arrangements of the 3 letters c, a, t. This is 3!, the factorial of 3.

That leads us to a rule.

> If n is the number of objects in a set, then the number of permutations for that set is $n!$.

Practice

Find the **number of permutations** *for each of the following.*

Remember: If n is the number of objects in a set, then the number of permutations for that set is $n!$.

1. {d, o, g}

   Answer: _____

2. {1, 2, 3, 4}

   Answer: _____

3. {g, o, a, t, s}

   Answer: _____

4. {f, r, i, e, n, d}

   Answer: _____

5. {c, a, r, s}

   Answer: _____

6. {5, 10, 15, 20, 25, 30}

   Answer: _____

7. 5 objects

   Answer: _____

8. 6 objects

   Answer: _____

9. 7 objects

   Answer: _____

10. 10 objects

    Answer: _____

Permutations of Only Some Elements in a Set

Sometimes, we want to arrange only *some* of the elements in a set. As you might guess, this will require a slightly different process. We may want to find the number of permutations for 5 objects if we only use 3 of the objects at a time. If n still represents the number of objects in the set and r represents the number of objects we want to include, we use the following *formula.*

number of permutations
$_nP_r = \frac{n!}{(n-r)!}$

In this case, $_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \bullet 4 \bullet 3 \bullet \cancel{2} \bullet \cancel{1}}{\cancel{2} \bullet \cancel{1}} = 60.$

Try the following in the next practice.

Practice

Use the **formula** *below to* **evaluate** *each of the following.*

number of permutations

$${}_nP_r = \frac{n!}{(n-r)!}$$

1. ${}_5P_2$

   Answer: _____

2. ${}_6P_3$

   Answer: _____

3. ${}_6P_2$

   Answer: _____

4. ${}_4P_3$

   Answer: _____

5. ${}_5P_4$

   Answer: _____

6. ${}_4P_2$

   Answer: _____

Find the **number of permutations** *of* **8 objects** *taken*

7. 1 at a time.

   Answer: _____

8. 2 at a time.

   Answer: _____

9. 3 at a time.

   Answer: _____

10. 4 at a time.

    Answer: _____

11. 5 at a time.

    Answer: _____

12. 6 at a time.

    Answer: _____

13. 7 at a time.

    Answer: _____

Practice

Use the **formula** *below to find* ${}_nP_r$ *for the* **given values** *of n and r.*

number of permutations

$${}_nP_r = \frac{n!}{(n-r)!}$$

1. $n = 11, r = 8$

   Answer: _____

2. $n = 6, r = 4$

   Answer: _____

3. $n = 9, r = 1$

   Answer: _____

4. $n = 10, r = 3$

   Answer: _____

How many ways can you **park 7 different cars** *in a lot if there are* **only spaces** *for*

5. 6 cars?

   Answer: _____

6. 4 cars?

   Answer: _____

7. 5 cars?

   Answer: _____

Permutations for Sets with Repeated Elements

Sometimes we want to find permutations for sets that have *repeated* elements. For instance, we may want to find the number of ways the letters in the word SOCCER can be arranged. Because there are two Cs in the word, that reduces the number of unique, or different, arrangements we could have. When this happens, we let n_1 represent the number of times an element is repeated and use another formula.

number of permutations with one set of repeated letters

$$P = \frac{n!}{n_1!}$$

So, for SOCCER, we let $n = 6$ because there are 6 letters in the word. We let $n_1 = 2$, because there are two Cs in SOCCER. So, to put it all together, we get the following.

$$P = \frac{6!}{2!} = \frac{6 \bullet 5 \bullet 4 \bullet 3 \bullet \cancel{2} \bullet \cancel{1}}{\cancel{2} \bullet \cancel{1}} = 360$$

Suppose we want to find the number of ways the letters in the word TWEETER can be arranged. Notice that there are 2 repeated letters. Our formula will now look like the following.

number of permutations with two sets of repeated letters

$$P = \frac{n!}{n_1!n_2!}$$

We let $n = 7$ because there are 7 letters in the word. We let $n_1 = 2$, because there are 2 Ts and $n_2 = 3$ because there are 3 Es in TWEETER. Now let's calculate.

$$P = \frac{7!}{2!3!} = \frac{7 \bullet 6 \bullet 5 \bullet \cancel{4}^{2} \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}}{\cancel{2} \bullet \cancel{1} \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}} = 420$$

Now it's your turn to try some!

Practice

Use the **formulas** *below to find the* **number of ways** *the letters in each word can be arranged.*

number of permutations with one set of repeated letters	**number of permutations with two sets of repeated numbers**
$P = \frac{n!}{n_1!}$	$P = \frac{n!}{n_1!n_2!}$

1. RUNNING

   Answer: _____

2. VOLLEYBALL

   Answer: _____

3. COOKIE

   Answer: _____

4. NOON

   Answer: _____

5. MIDDLE

   Answer: _____

6. FOOTBALL

   Answer: _____

7. SCHOOLS

   Answer: _____

8. HUSHPUPPY

   Answer: _____

Combinations

Combinations are very similar to permutations. They both refer to arrangements of the elements in a set. With *combinations*, however, we use the concept of **subsets** to help determine the results.

Set B is considered to be a *subset* of Set A if each element of Set B is also an element of Set A.

For example,

if Set A = {c, a, t} and Set B = {a, t}, then Set B is a subset of Set A.

In fact {c, a, t} has several subsets. Let's list them. This is much easier if you use some system to insure that you get them all.

The subsets of {c, a, t} are
{c}
{a}
{t}
{c, a}
{c, t}
{a, t}
{c, a, t}
{}

Note:

1. Set A is a subset of itself and the **empty set**, or **null set (ø)**, is also a subset of itself.

2. {a, t} is considered to be the same as {t, a} and therefore *not* counted twice.

Therefore, Set A has 8 subsets. Do you suppose that every set with 3 elements will have 8 subsets? Yes. In fact you can find the number of subsets for any set by using a simple fact.

How to Find the Number of Subsets for Any Set

The number of subsets =	2^n, when n represents the number of elements in the original set.

So, if Set P = {4, 5, 6, 7, 8}, the number of subsets for Set $P = 2^5 = 32$.

Practice

Find the **number of subsets** *for each of the following sets.*

1. {a, b, c} _____
2. {w, x, y, z} _____
3. {5, 6} _____
4. {f, r, o, g, s} _____

Check Yourself:

If the sum of all your answers for numbers 1-4 above equals 60, then you did them correctly! If not, reread the lesson and try again.

Formula for Combinations

Because subsets with the same elements in different orders are considered to be the same subset, and combinations are based on the concept of subsets, you can see that there will generally be fewer combinations than permutations. One way to make a number *smaller* is to make the *denominator* larger. With that in mind, look at the formula for combinations.

formula for combinations

$$ {}_nC_r = \frac{n!}{r!(n-r)!} $$

Notice that the formula for combinations is very similar to the one for permutations. The only change is that the denominator *increases* because we multiply it by $r!$.

Let's see an example.

$$ {}_4C_3 = \frac{4!}{3!(4-3)!} = $$

Work in the parentheses first.

$$ \frac{4 \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}}{\cancel{3} \bullet \cancel{2} \bullet \cancel{1} \bullet (1)} = $$

$$ \frac{4}{1} = $$

$$ 4 $$

Now it's your turn.

Practice

Use the **formula** *below to* **evaluate** *each of the following.*

formula for combinations

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

1. ${}_{10}C_6$

   Answer: _____

2. ${}_6C_2$

   Answer: _____

3. ${}_5C_2$

   Answer: _____

4. ${}_{7}C_{3}$

Answer: _____

5. ${}_{11}C_{9}$

Answer: _____

6. ${}_{20}C_{4}$

Answer: _____

Practice

How many **combinations** *can be formed from the letters in NUMBERS taking them*

1. 5 at a time? _____
2. 4 at a time? _____
3. 3 at a time? _____
4. 2 at a time? _____

Calculate *each of the following.*

5. How many combinations of 6 people can be chosen from a group of 10 to be in a photograph?

   Answer: _____

6. How many combinations of 5 players can be chosen from 12 students who try out for the basketball team?

   Answer: _____

7. How many combinations of 4 members can be chosen for the debate team if 8 students try out?

   Answer: _____

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. a list of data organized to show how many times each item or event occurs

_____ 2. information in the form of numbers gathered for statistical purposes

_____ 3. the score or data point found most often in a set of numbers

_____ 4. a set whose elements are taken from a larger set

_____ 5. a bar graph that shows the frequency of data within intervals

_____ 6. a collection of distinct objects or numbers

_____ 7. a measure of the likelihood that a given event will occur; expressed as a ratio of one event occurring (favorable outcome) to the number of equally likely possible outcomes

_____ 8. the median of the upper 50 percent of a set of data

_____ 9. the arithmetic average of a set of numbers

_____ 10. a numerical operation in which a number is multiplied by all positive integers less than that number

_____ 11. two events in which the first affects the outcome of the second event

A. data

B. dependent events

C. factorial (!)

D. frequency distribution

E. histogram

F. mean (or average)

G. mode

H. probability

I. set

J. subset

K. upper quartile

Practice

Match each definition with the correct term. Write the letter on the line provided.

_____ 1. the mean, median, and mode of a set of data

_____ 2. two events in which the outcome of the first event does *not* affect the outcome of the second event

_____ 3. a graph that organizes data by place value to compare data frequencies

_____ 4. the middle point of a set of rank-ordered numbers where half of the numbers are above the median and half are below it

_____ 5. a possible result or outcome in probability

_____ 6. an arrangement, or listing, of objects or events in which order is not accounted for

_____ 7. the median of the lower 50 percent of a set of data

_____ 8. a basic graphing tool that displays centering, spread, and distribution of a data set

_____ 9. an arrangement, or listing, of objects or events in which order is important

_____ 10. the lowest value (L) in a set of numbers through the highest value (H) in the set

_____ 11. events that cannot both take place at the same time

A. box-and-whisker plot (box plots)

B. combinations

C. event

D. independent events

E. lower quartile

F. measure of central tendency

G. median

H. mutually exclusive events

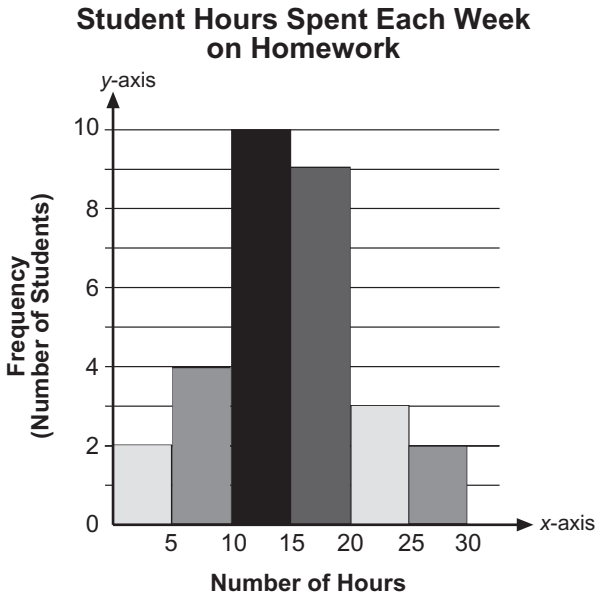
I. permutation

J. range (of a set of numbers)

K. stem-and-leaf plot

Unit Review

Answer the following.



1. Use the *histogram* to answer the following question. How many students studied between 20 and 25 hours per week?

   Answer: _____

2. Draw a *stem-and-leaf plot* for the following set of data.

   35, 42, 45, 44, 68, 54, 49, 61, 42

Stem	Leaves

3. Find the *mean* for the following set of numbers.

   33, 53, 26, 64, 29.8, 56, 24, 61

   mean: _____

4. Find the *median* for the following set of numbers.

   23, 54, 24, 68, 19.8, 35, 24, 65

   median: _____

5. Find the *mode* for the following set of numbers.

   43, 34, 24, 66, 29.8, 36, 24, 65

   mode: _____

6. Find the *range* for the following set of numbers.

   33, 54, 24, 65, 29.8, 36, 24, 67

   range: _____

7. Draw a *box-and-whisker plot* for the following set of data.

25, 31, 43, 47, 54, 59, 65

Q_1 = _____

Q_2 (median) = _____

Q_3 = _____

8. How many *odd* 2-digit positive integers less than 60 are there?

Remember: Using the counting principle, if a first *event* has *n* outcomes and a second event has *m* outcomes, then the first event followed by the second event has *n* x *m* outcomes.

Answer: _____

☐ x ☐

9. In how many ways can a 6-question True/False quiz be answered if every question *must be answered*?

Answer: _____

☐ x ☐ x ☐ x ☐ x ☐ x ☐

10. How many *odd* positive integers *less than* 400 can be written using the numbers 3, 4, 5, 6, and 7?

Answer: _____

☐ x ☐ x ☐

11. A restaurant serves 5 main dishes, 3 types of salad, and 4 different desserts. How many *different meal choices* could you make by choosing one main dish, one salad, and one dessert?

Answer: _____

12. When flipping 1 regular coin, what is the probability of the coin landing *heads up*?

Remember: Probability = $\frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$

Answer: _____

13. When flipping 2 regular coins, what is the probability of the coins landing with 1 head up *and* 1 tail up?

Answer: _____

Coin 1	Coin 2

14. When tossing 3 coins, find the probability that you will toss 2 heads *and* 1 tail.

Hint: Draw a possible outcome chart.

Answer: _____

15. When tossing 2 dice, find the probability that *exactly 1* die shows a 4.

Answer: _____

Possible Outcomes of Rolling 2 Dice

16. A bag of marbles contains

- 4 yellow
- 3 green
- 6 orange
- 7 blue marbles.

What is the probability of drawing 2 orange marbles if you *draw once* and *replace* the marble, and *then draw again*?

Hint: Determine if the event is *independent* or *dependent*.

Answer: _____

17. 3!4!

Example: 4!5! = 4 x 3 x 2 x 1 x 5 x 4 x 3 x 2 x 1 = 2,880

Answer: _____

18. Find the *number of permutations* for {f, r, o, g}.

Remember: If *n* is the number of objects in a set, then the number of permutations for that set is *n*!.

Answer: _____

19. Evaluate ${}_5P_3$ using the following formula.

number of permutations

$${}_nP_r = \frac{n!}{(n-r)!}$$

Answer: _____

20. Find the number of permutations of 8 objects *taken* 2 at a time. Use the same formula as in number 19.

Answer: _____

21. Find the number of ways the letters in HOUSE can be *arranged*. Use the following formula.

number of permutations with one set of repeated letters

$$P = \frac{n!}{n_1!}$$

Answer: _____

22. Find the number of ways the letters in PARALLEL can be *arranged*. Use the same formula as in number 21.

Answer: _____

23. Find the *number of subsets* for {a, b, c, d}.

Remember: The number of subsets = 2^n, when n represents the number of elements in the original set.

Answer: _____

24. Evaluate ${}_9C_6$ using the following formula.

formula for combinations

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

Answer: _____

25. How many *combinations* of 6 players can be chosen from 14 students who try out for the volleyball team? Use the same formula as in number 24.

Answer: _____

Appendices

Table of Squares and Approximate Square Roots

n	n^2	$\sqrt{n}$
1	1	1.000
2	4	1.414
3	9	1.732
4	16	2.000
5	25	2.236
6	36	2.449
7	49	2.646
8	64	2.828
9	81	3.000
10	100	3.162
11	121	3.317
12	144	3.464
13	169	3.606
14	196	3.742
15	225	3.873
16	256	4.000
17	289	4.123
18	324	4.243
19	361	4.359
20	400	4.472
21	441	4.583
22	484	4.690
23	529	4.796
24	576	4.899
25	625	5.00
26	676	5.099
27	729	5.196
28	784	5.292
29	841	5.385
30	900	5.477
31	961	5.568
32	1,024	5.657
33	1,089	5.745
34	1,156	5.831
35	1,225	5.916
36	1,296	6.000
37	1,369	6.083
38	1,444	6.164
39	1,521	6.245
40	1,600	6.325
41	1,681	6.403
42	1,764	6.481
43	1,849	6.557
44	1,936	6.633
45	2,025	6.708
46	2,116	6.782
47	2,209	6.856
48	2,304	6.928
49	2,401	7.000
50	2,500	7.071

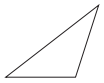
n	n^2	$\sqrt{n}$
51	2,601	7.141
52	2,704	7.211
53	2,809	7.280
54	2,916	7.348
55	3,025	7.416
56	3,136	7.483
57	3,249	7.550
58	3,364	7.616
59	3,481	7.681
60	3,600	7.746
61	3,721	7.810
62	3,844	7.874
63	3,969	7.937
64	4,096	8.000
65	4,225	8.062
66	4,356	8.124
67	4,489	8.185
68	4,624	8.246
69	4,761	8.307
70	4,900	8.367
71	5,041	8.426
72	5,184	8.485
73	5,329	8.544
74	5,476	8.602
75	5,625	8.660
76	5,776	8.718
77	5,929	8.775
78	6,084	8.832
79	6,241	8.888
80	6,400	8.944
81	6,561	9.000
82	6,724	9.055
83	6,889	9.110
84	7,056	9.165
85	7,225	9.220
86	7,396	9.274
87	7,569	9.327
88	7,744	9.381
89	7,921	9.434
90	8,100	9.487
91	8,281	9.539
92	8,464	9.592
93	8,649	9.644
94	8,836	9.695
95	9,025	9.747
96	9,216	9.798
97	9,409	9.849
98	9,604	9.899
99	9,801	9.950
100	10,000	10.000

Mathematical Symbols

$\div$ *or* /	divide	$\parallel$	is parallel to
x *or* $\bullet$	times	$\approx$	is approximately equal to
$=$	is equal to	$\cong$	is congruent to
$-$	negative	$\sim$	is similar to
$+$	positive	$\sqrt{}$	nonnegative square root
$\pm$	positive or negative	%	percent
$\neq$	is not equal to	π	pi
$>$	is greater than	$\overleftrightarrow{AB}$	line *AB*
$<$	is less than	$\overline{AB}$	line segment *AB*
$\not>$	is not greater than	$\overrightarrow{AB}$	ray *AB*
$\not<$	is not less than	$\triangle ABC$	triangle *ABC*
$\geq$	is greater than or equal to	$\angle ABC$	angle *ABC*
$\leq$	is less than or equal to	$m\overline{AB}$	measure of line segment *AB*
$^\circ$	degrees	$m\angle ABC$	measure of angle *ABC*
$\perp$	is perpendicular to		

FCAT Mathematics Reference Sheet

Formulas

	**triangle**	$A = \frac{1}{2}bh$
	**rectangle**	$A = lw$
	**trapezoid**	$A = \frac{1}{2}h(b_1 + b_2)$
	**parallelogram**	$A = bh$
	**circle**	$A = \pi r^2$

Key	
b =	base
h =	height
l =	length
w =	width
ℓ =	slant height
$S.A.$ =	surface area
d =	diameter
r =	radius
A =	area
C =	circumference
V =	volume
Use 3.14 or $\frac{22}{7}$ for π.	

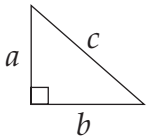
circumference
$C = \pi d$ or $C = 2\pi r$

		Volume	Total Surface Area		
	**right circular cone**	$V = \frac{1}{3}\pi r^2 h$	$S.A. = \frac{1}{2}(2\pi r)\ell + \pi r^2$	or	$S.A. = \pi r\ell + \pi r^2$
	**square pyramid**	$V = \frac{1}{3}lwh$	$S.A. = 4(\frac{1}{2}l\ell) + l^2$	or	$S.A. = 2l\ell + l^2$
	**sphere**	$V = \frac{4}{3}\pi r^3$	$S.A. = 4\pi r^2$		
	**right circular cylinder**	$V = \pi r^2 h$	$S.A. = 2\pi rh + 2\pi r^2$		
	**rectangular solid**	$V = lwh$	$S.A. = 2(lw) + 2(hw) + 2(lh)$		

In the following formulas, n represents the number or sides.

- In a polygon, the sum of the measures of the interior angles is equal to $180(n - 2)$.
- In a regular polygon, the measure of an interior angle is equal to $\frac{180(n-2)}{n}$.

FCAT Mathematics Reference Sheet

Pythagorean theorem:  $a^2 + b^2 = c^2$	Distance between two points $P_1\ (x_1, y_1)$ and $P_2\ (x_2, y_2)$: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Slope-intercept form of an equation of a line: $y = mx + b$ where m = slope and b = the y-intercept.	Midpoint between two points $P_1\ (x_1, y_1)$ and $P_2\ (x_2, y_2)$: $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$
Distance, rate, time formula: $d = rt$ where d = distance, r = rate, t = time.	Simple interest formula: $I = prt$ where p = principal, r = rate, t = time.

Conversions

1 yard = 3 feet = 36 inches
1 mile = 1,760 yards = 5,280 feet
1 acre = 43,560 square feet
1 hour = 60 minutes
1 minute = 60 seconds

1 cup = 8 fluid ounces
1 pint = 2 cups
1 quart = 2 pints
1 gallon = 4 quarts

1 liter = 1000 milliliters = 1000 cubic centimeters
1 meter = 100 centimeters = 1000 millimeters
1 kilometer = 1000 meters
1 gram = 1000 milligrams
1 kilogram = 1000 gram

1 pound = 16 ounces
1 ton = 2,000 pounds

Metric numbers with four digits are presented without a comma (e.g., 9960 kilometers). For metric numbers greater than four digits, a space is used instead of a comma (e.g., 12 500 liters).

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Production Software

Adobe PageMaker 6.5. Mountain View, CA: Adobe Systems.

Adobe Photoshop 5.0. Mountain View, CA: Adobe Systems.

Macromedia Freehand 8.0. San Francisco: Macromedia.

Microsoft Office 98. Redmond, WA: Microsoft.